Module - 2

2.1 Friction

Whenever a body moves or tends to move over another body, a force opposite to the direction in which the body moves or tends to move is developed at the contact surfaces. This force is called frictional force or simply friction. The main cause of friction is the surface roughness. If the contact surfaces are perfectly smooth, the frictional force will be zero. Frictional force has the remarkable property of adjusting itself in magnitude to the force tending the motion. When the force tending the motion increases the frictional force increases. But there is a limit beyond which this frictional force cannot be increased. This maximum value of frictional force is known as limiting friction which is the frictional force when motion is about to start. When the applied force on the body is greater than the limiting friction, the body moves in the direction of applied force and when the applied force is less than the limiting friction, the body remains at rest.

Coefficient of friction

It is experimentally found that the magnitude of limiting friction bears a constant ratio to the normal reaction between the two surfaces.ie., the limiting friction is proportional to the normal reaction at the contact surface. This constant of proportionality is called coefficient of friction.

Limiting friction, F_{max}, is proportional to the normal reaction, R_N.

$$F_{max} \alpha\ R_N$$

$$F_{max} = \mu R_N$$

$$\mu = \frac{F_{max}}{R_N},$$ μ is the coefficient of friction at the contact surface.

The coefficient of friction when the body is at rest is called coefficient of static friction. It is denoted by μ_s. The coeffiecient of friction when the body is in motion is called coefficient of dynamic friction or coefficient of kinetic friction. It is denoted by μ_d or μ_k. The value of μ_k is slightly less than μ_s but for the most of the engineering applications μ_d is taken as μ_s and then $\mu_d = \mu_s = \mu$

Angle of friction

It is the angle between the normal reaction at the contact surface and the resultant of normal reaction and limiting friction. It is denoted by ϕ

$$\tan \phi = \frac{\mu R_N}{R_N} = \mu$$

$$\text{Angle of friction } \phi = \tan^{-1} \mu$$

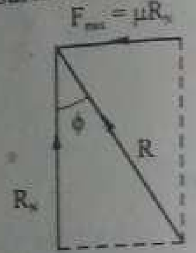

Fig. 2.1

Angle of repose

It is the maximum inclination of a plane on which a body can remains at rest, without applying external force.

When motion impends, frictional force $F = \mu \times R_N$

Resolving the forces along the inclined plane,

$$\mu R_N - W \sin \alpha = 0$$

$$\mu R_N = W \sin \alpha$$

Resolving the forces perpendicular to the inclined plane,

$$R_N - W \cos \alpha = 0$$

$$R_N = W \cos \alpha$$

$$\therefore \mu W \cos \alpha = W \sin \alpha$$

$$\tan \alpha = \mu = \tan \phi$$

$$\therefore \alpha = \phi$$

Angle of repose is equal to angle of friction.

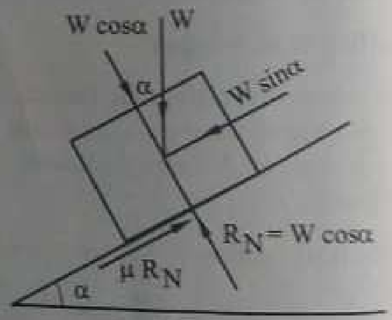

Fig. 2.2

Cone of friction.

Consider the limiting equilibrium of a body kept on a horizontal surface. Let P be the applied force and R_N be the normal reaction. The frictional force $F = \mu \times R_N$. The resultant of limiting friction and normal reaction makes an angle equal to angle of friction with the normal reaction. When the direction of external force is changed the direction of resultant changes but the angle between the normal reaction and the resultant force will be the same.

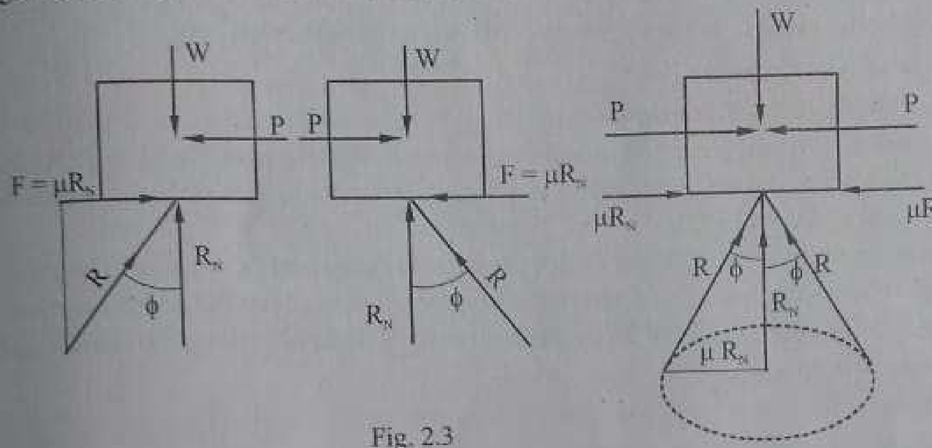

Fig. 2.3

Thus when the direction of external force is gradually changed through 360^0, the resultant R generates a right circular cone with semi cone angle equal to ϕ. This cone is called friction cone or cone of friction. The axis of this cone will be the normal reaction and the generators are the resultant force and base radius is equal to the limiting frictional force.

2.2. Sliding friction

It is the friction experienced by a body when it slides or tends to slide over another body. When unlubricated (dry) surfaces of two bodies in contact slides or tends to slide, the resulting friction is called dry friction. The principles of dry friction were developed largely from the experiments of Coulomb in 1781 and hence dry fiction is also called Coulomb friction. Consider a body of weight W kept on a horizontal surface as shown in Fig. 2.4. P is the applied force on the body, F is the frictional force and R_N is the normal reaction at the contact surface.

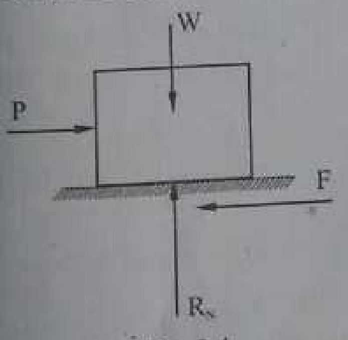

Fig. 2.4

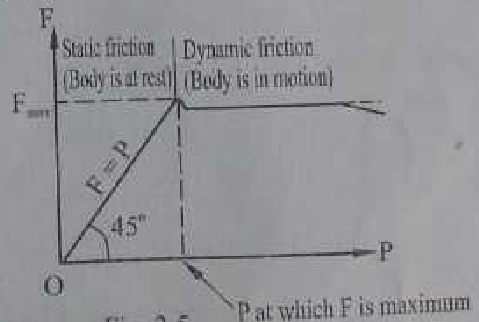

Fig. 2.5

Fig. 2.5 shows the variation of frictional force F with the applied force P. When the applied force P is zero , frictional force F is also zero. For certain value of P, the the frictional force is maximum. From zero to this value of P ,the body remains at rest and hence for equilibrium of the body F = P. At this value of P the body is at the verge of motion. Further increase of P causes a small and sudden decrease of F and then remains constant. At very high velocity of the body there is a small decrease in the frictional force F. Refer Fig. 2.5

When the body is at rest, $F = P < F_{max}$

When the body is in limiting equilibrium (at the verge of motion), $F = F_{max}$

When the body is in motion $P > F_{max}$

2.3. Coulomb's laws of friction.

Coulomb in 1781 presented certain conclusions which are applicable when the body is in limiting equilibrium and when the body is in motion. These conclusions are known as coulomb's laws of friction. These laws are :

1. The maximum friction that can be developed is independent of the area of contact.
2. At low velocity the frictional force is independent of velocity of the contact surface.
3. The maximum frictional force is proportional to the normal reaction at the contact surface.

Other laws of friction are :

1. The force of friction always acts in a direction opposite to the direction in which the body moves or tends to move.
2. Till the limiting value is reached the magnitude of friction is equal to the external force which tends to move the body
3. The force of friction depends on the roughness of the surfaces in contact. When two perfectly smooth surfaces are in contact, the frictional force is zero.

2.4. Analysis of single bodies.

When a single body slides or tends to slide over another single surface, the various forces acting at the contact surface and the external forces acting on the body including the self weight of the body can be resolved horizontally and vertically or along any two mutually perpendicular directions. Applying the conditions for equilibrium we can solve for the unknowns in the problem.

Example 2.1.

A body of weight 1000 N is kept on a horizontal rough surface. Coefficient of static friction is 0.2 and coefficient of dynamic friction is 0.18. Calculate the frictional force when the applied force P as shown in Fig. 2.6 is (i) 0, (ii) 100 N, (iii) 200 N and (iv) 300 N

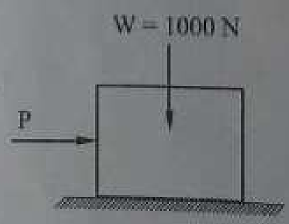

Fig. 2.6

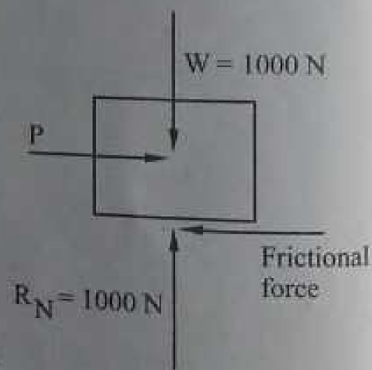

Fig. 2.7

Solution

(i) When P = 0

The body has no tendency to move in the horizontal direction and hence the frictional force is zero.

(ii) When P is 100 N

The limiting Friction $F = \mu \times R_N$

$= 0.2 \times 1000$

$= 200$ N

Since the force tending the motion is less than the limiting friction the body does not move. But there is a tendency to move the body towards right.

The frictional force = applied force = 100 N.

(iii) When P = 200 N

The motion impends. The body is in limiting equilibrium and hence the frictional force

$$F = \mu \times R_N = 0.2 \times 1000 = 200 \text{ N.}$$

(iv) When P = 300 N

Since the limiting friction is 200 N and the applied force is 300 N, the body moves towards right. Frictional force $= \mu_k \times R_N = 0.18 \times 1000 = 180$ N

Example 2.2.

A body of weight 90 N is placed on a rough horizontal plane. Determine the coefficient of friction if a horizontal force of 63 N just cause the body to slide over the horizontal plane.

Solution.

Since there is no motion in the vertical direction, net force in the vertical direction is zero.

$$\therefore R_N - W = 0$$

$$R_N = W = 90 \text{ N}$$

Since P just causes motion,

$$\mu R_N = P = 63 \text{ N}$$

$$\therefore \mu = \frac{63}{R_N} = \frac{63}{90} = 0.7$$

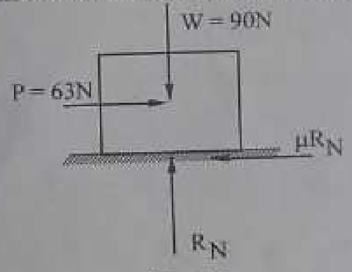

Fig. 2.8

Example 2.3.

A block of weight 200 N is placed on a rough horizontal floor. If $\mu = 0.25$, find the pull P required to move the block if P is inclined upwards at 30^0 to the horizontal

Solution

Since there is no motion in the vertical direction, net force in the vertical direction is zero.

$$R_N + P \sin 30 - 200 = 0$$

$$R_N = 200 - P \sin 30$$

Since P is the limiting force causing motion of the block,

$$P \cos 30 - \mu R_N = 0$$

$$P \cos 30 = 0.25\,(200 - P \sin 30)$$

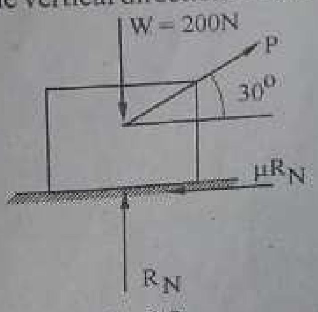

Fig. 2.9

$$= 50 - 0.25\ P \sin 30$$

$$P (\cos 30 + 0.25 \sin 30) = 50$$

$$P = 50.45\ N$$

Example 2.4 | KTU Jan 2017 |

A force of 200 N is required just to move a body up an inclined plane of angle 15^0 when the force is applied parallel to the plane. When the inclination of the plane is made 20^0, the force required, again applied parallel to the plane is found to be 230 N. Find the weight of the body and coefficient of friction.

Solution.

In both cases the body tends to move up the inclined plane. Therefore the direction of frictional force will be along the plane in the downward direction. Consider the equilibrium of body in the first case,

Resolving the forces normal to the plane,

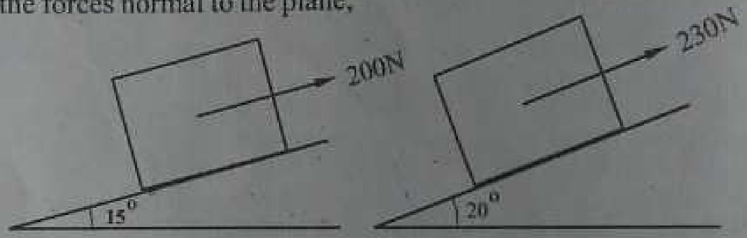

Fig. 2.10

$$R_{N1} - W \cos 15 = 0$$

$$R_{N1} = W \cos 15 \text{ ----(i)}$$

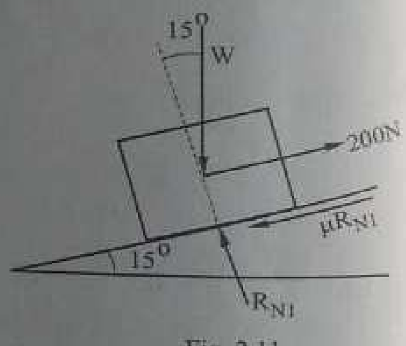

Fig. 2.11

Resolving the forces along the inclined plane

$$200 - \mu R_{N1} - W \sin 15 = 0$$

$$200 - \mu W \cos 15 - W \sin 15 = 0$$

$$W (\sin 15 + \mu \cos 15) = 200 \text{ ----(ii)}$$

Consider the equilibrium of the body in the second case, resolving the forces normal to the plane,

$$R_{N2} - W \cos 20 = 0$$

$$R_{N2} = W \cos 20$$

Resolving the forces along the inclined plane,

$$230 - \mu R_{N2} - W \sin 20 = 0$$

$$230 - \mu W \cos 20 - W \sin 20 = 0$$

$$W (\sin 20 + \mu \cos 20) = 230 \text{ -----(iii)}$$

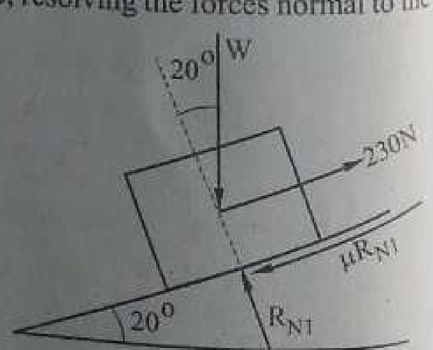

Fig. 2.12

From equations (ii) and (iii),

$$\frac{\sin 20 + \mu \cos 20}{\sin 15 + \mu \cos 15} = \frac{230}{200} = 1.15$$

$$\sin 20 + \mu \cos 20 = 1.15 \sin 15 + 1.15 \mu \cos 15$$

$$\mu\ (1.15 \cos 15 - \cos 20) = \sin 20 - 1.15 \sin 15$$

$$\mu = \frac{\sin 20 - 1.15 \sin 15}{1.15 \cos 15 - \cos 20} = 0.26$$

Example 2.5.

A uniform wooden cube of side 1m and weighing 5000 N rests on its side on a horizontal plane. Find what maximum horizontal force can be applied at the top edge of the cube to just make it slide without overturning. The coefficient of friction is 0.25.

Solution

Given, W = 5000N, a = 1m, η=0.25

Let P be the horizontal force required to just slide the block.

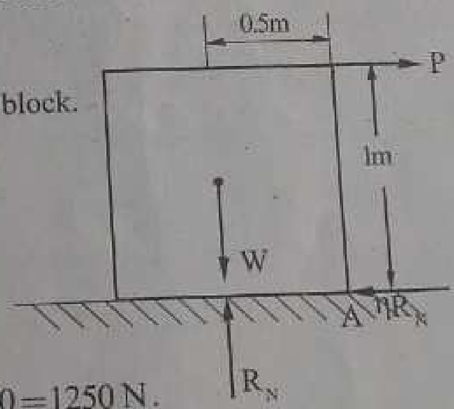

Fig. 2.13

The limiting frictional force is $\eta R_N = 0.25 \times R_N$

For $\Sigma F_V = 0$

$$R_N - W = 0$$

$$R_N = W.$$

Frictional force $= \eta R_N = \eta \times W = 0.25 \times 5000 = 1250$ N.

In the limiting equilibrium, For $\Sigma F_H = 0$,

$$P - \eta R_N = 0$$

$$P = \eta R_N = 1250 \text{ N}.$$

To calculate the minimum force required to overturn the block.

When the block just overturns about an edge, the contact between the block and the horizontal plane brakes and hence the normal reaction R_N is zero.

For $\Sigma M = 0$, taking moments about A,

$$P \times 1 - W \times 0.5 = 0$$

$$P = W \times 0.5 = 5000 \times 0.5$$

$$= 2500\text{N}$$

If we apply a force which is more than 1250 N and less than 2500 N, the block will sid without overturning.

2.5. Wedge friction.

Wedges are small pieces of materials with triangular or trapezoidal cross section. They are generally used for lifting heavy weights, for slight adjustments in the position of a body etc. The weight of the wedge is very small compared to the weight lifted. Hence generally

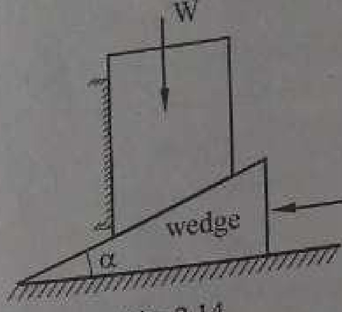

Fig. 2.14

the weight of wedge will be neglected. The problems on wedges are generally the problems of equalibrium of bodies on inclined planes.

Fig 2.15 shows the forces acting on the body.

Consider the equilibrium of the body,

Resolving the forces vertically,

$R_2 \cos\alpha - \mu_2 R_2 \sin\alpha - \mu_1 R_1 - W = 0$

Resolving forces horizontally,

$$R_1 - \mu_2 R_2 \cos\alpha - R_2 \sin\alpha = 0$$

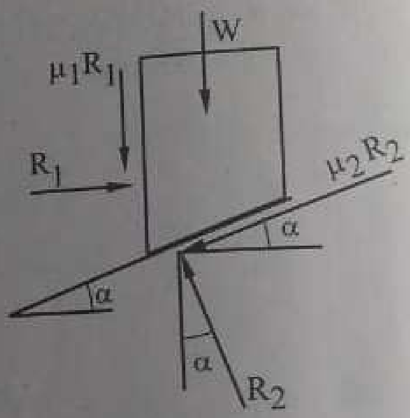

Fig. 2.15

For equlibrium of the wedge

Resolving the forces horizontally,

$\mu_3 R_3 + \mu_2 R_2 \cos\alpha + R_2 \sin\alpha - P = 0$

Resolving the forces vertically,

$$R_3 + \mu_2 R_2 \sin\alpha - R_2 \cos\alpha = 0.$$

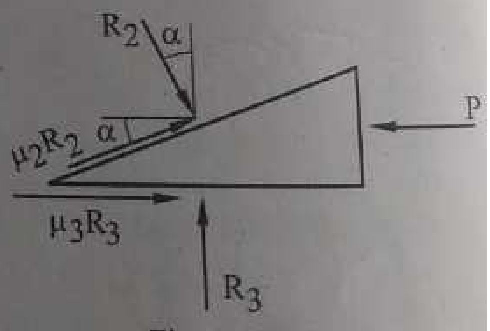

Fig. 2.16

Example 2.6.

Find the horizontal force P on the 10° wedge shown in Fig. 2.17 to raise the 1500 N l
The coefficient of friction is 0.3 at all contact surfaces.

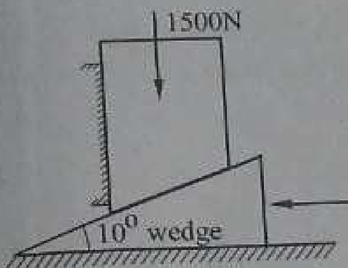

Fig. 2.17

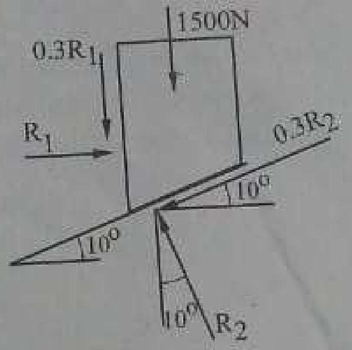

Fig. 2.18

Solution.

Consider the equilibrium of the load of 1500 N.

Resolving the forces horizontally

$$R_1 - 0.3\ R_2\ \cos 10 - R_2 \sin 10 = 0$$

$$R_1 = R_2\ (0.3\ \cos 10 + \sin 10\ )$$

$$= 0.47\ R_2 \quad \text{------(i)}$$

Resolving the forces vertically

$$R_2 \cos 10 - 0.3\ R_2 \sin 10 - 1500 - 0.3\ R_1 = 0$$

$$R_2\ \cos 10 - 0.3\ R_2\ \sin 10\ - 0.3\ (0.47\ R_2\ ) = 1500$$

$$R_2\ (\cos 10 - 0.3\ \sin 10 - 0.3 \times\ 0.47\ ) = 1500$$

$$R_2 = 1894.63\ N$$

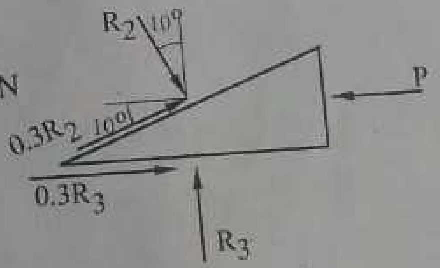

Fig. 2.19

Consider the equilibrium of wedge .

Resolving the forces vertically,

$$R_3 + 0.3\ R_2 \sin 10 - R_2 \cos 10 = 0$$

$$R_3 = R_2\ (\ \cos 10 - 0.3\ \sin 10)$$

$$= 1894.63\ (\ \cos 10 - 0.3\ \sin 10\ )$$

$$= 1767.15\ N$$

Resolving the forces horizontally

$$0.3\ R_2\ \cos 10 + R_2\ \sin 10 + 0.3\ R_3\ - P = 0$$

$$P = 1894.63\ (0.3\ \cos 10 + \sin 10\ ) + 0.3 \times 1767.15$$

$$= 1418.90\ N$$

Example 2.7.

Two wedges A and B are used to raise another block C weighing 1000 N as shown in Fig. 2.20. Assuming coefficient of friction as 0.25 for all the surfaces, determine the value of P for impending upwards motion of block C.

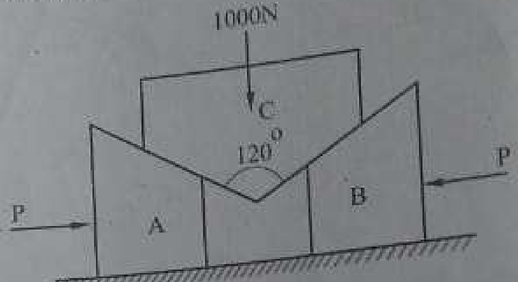

Fig. 2.20

Solution.

Because of symmetry, the reaction at the contact surface between A and C will be same as that between B and C. Let this reaction be R.

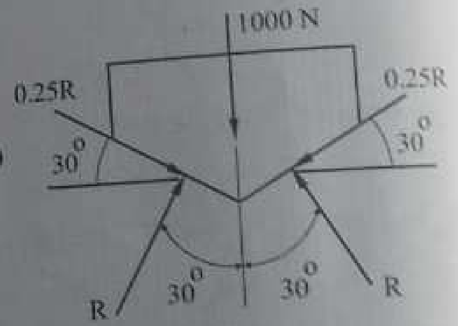

Fig. 2.21

Consider the equilibrium of block C.

Resolving the forces vertically,

$$R \cos 30 + R \cos 30 - 1000 - 2 \times 0.25 R \sin 30 = 0$$

$$2 R (\cos 30 - 0.25 \sin 30) = 1000$$

$$R = 674.74 \text{ N}$$

Consider the equilibrium of wedge A.

Resolving the forces vertically

$$R_1 + 0.25 R \sin 30 - R \cos 30 = 0$$

$$R_1 = R (\cos 30 - 0.25 \sin 30)$$

$$= 674.74 (\cos 30 - 0.25 \sin 30)$$

$$= 500 \text{ N.}$$

Resolving the forces horizontally,

$$P - 0.25 R_1 - 0.25 R \cos 30 - R \sin 30 = 0$$

$$P = 0.25 R_1 + R (0.25 \cos 30 + \sin 30)$$

$$= 0.25 \times 500 + 674.74 (0.25 \cos 30 + \sin 30)$$

$$= 608.46 \text{ N}$$

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Example 2.8.

Two blocks A and B are resting against a wall and a floor as shown in Fig 2.22. Find the range of value of force P applied to the lower block for which the system remains in equilibrium. Coefficient of friction is 0.25 at the floor and 0.3 at the wall and 0.2 between the blocks

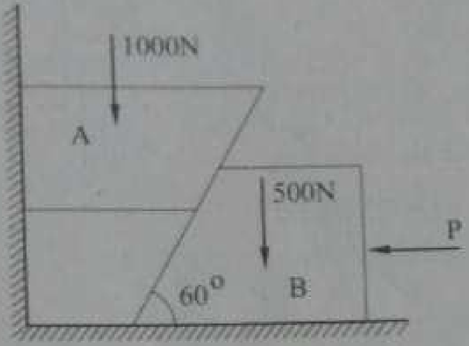


Fig. 2.22

Solution.

If the applied force is less than the minimum force required to keep the system in equilibrium, then the block A will move downwards and block B will move towards right. If the applied force is more than a certain value, then the block A will move upwards and block B will move towards left. The minimum value of P is the force required to prevent the block A from moving down and the maximum value of P is the force required just to move the block upwards.

Minimum value of force P

In this case the direction of frictional force at block A is upwards and the direction of frictional force at block B is towards left. Consider the equilibrium block A.

Resolving the forces vertically,

$$0.3\, R_1 + 0.2\, R_2 \sin 60 + R_2 \cos 60 - 1000 = 0$$

$$0.3\, R_1 + R_2\,(0.2 \sin 60 + \cos 60) = 1000$$

$$0.3\, R_1 + 0.673\, R_2 = 1000 \text{ ---(i)}$$

Resolving the forces horizontally,

$$R_1 + 0.2\, R_2 \cos 60 - R_2 \sin 60 = 0$$

$$R_1 = R_2\,(\sin 60 - 0.2 \cos 60)$$

$$= 0.766\, R_2.$$

Substituting this value of R_1 in eqn (i)

$$0.3\,(0.766\, R_2) + 0.673\, R_2 = 1000$$

$$0.903\ R_2 = 1000$$
$$R_2 = 1107.66\ N$$

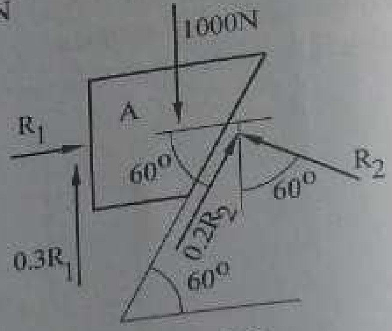


Fig. 2.23

Consider the equilibrium of block B

Resolving the forces vertically,

$$R_3 - 500 - R_2 \cos 60 - 0.2\ R_2 \sin 60 = 0$$
$$R_3 = 500 + R_2 (\cos 60 + 0.2 \sin 60)$$
$$= 500 + 1107.66\ (\cos 60 + 0.2 \sin 60)$$
$$= 1245.68\ N$$

Resolving the forces horizontally,

$$R_2 \sin 60 - 0.2\ R_2 \cos 60 - 0.25\ R_3 - P = 0$$
$$P = R_2 (\sin 60 - 0.2 \cos 60) - 0.25\ R_3$$
$$= 1107.66 \times 0.766 - 0.25 \times 1245.68$$
$$= 537.05\ N$$

Force P required to just move the block A upwards

Consider the equilibrium of block A

Resolving the forces horizontally,

$$R_1 - 0.2\ R_2 \cos 60 - R_2 \sin 60 = 0$$
$$R_1 = R_2 (0.2 \cos 60 + \sin 60) = 0.966\ R_2$$

Resolving the forces vertically,

$$R_2 \cos 60 - 0.2\ R_2 \sin 60 - 0.3\ R_1 - 1000 = 0$$
$$R_2 (\cos 60 - 0.2 \sin 60) - 0.3\ (0.966\ R_2) = 1000$$
$$R_2 \times 0.327 - 0.29\ R_2 = 1000$$
$$R_2 = 27027.03\ N$$

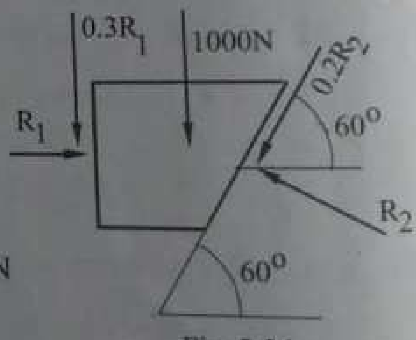


Fig. 2.24

Consider the equilibrium of block B.

Resolving the forces vertically,

$$R_3 + 0.2\ R_2 \sin 60 - R_2 \cos 60 - 500 = 0$$

$R_3 = R_2 (\cos 60 - 0.2 \sin 60) + 500$

$= 27027.03 \times 0.327 + 500$

$= 9337.84$ N

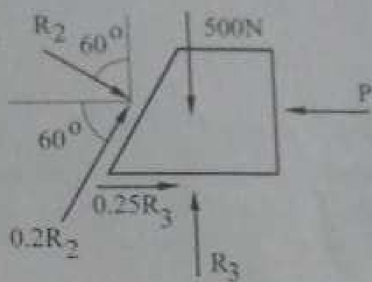


Fig. 2.25

Resolving the forces horizontally,

$R_2 \sin 60 + 0.2 R_2 \cos 60 + 0.25 R_3 - P = 0$

$P = 0.25 R_3 + 0.2 R_2 \cos 60 + R_2 \sin 60$

$= 0.25 \times 9337.84 + 27027.03 (0.2 \cos 60 + \sin 60)$

$P = 28443.26$ N

2.6. Ladder friction.

Fig 2.26 shows a ladder, AB, with the end A on the ground and end B on the wall. The ladder exerts a force on the wall and R_W is its reaction on the ladder. Similarly the ladder exerts a force on the ground and R_f is its reaction on the ladder. The upper end B of the ladder tends to slip downwards and hence the force of friction will be vertically upwards. The lower end tends to move away from the wall and hence the direction of friction force

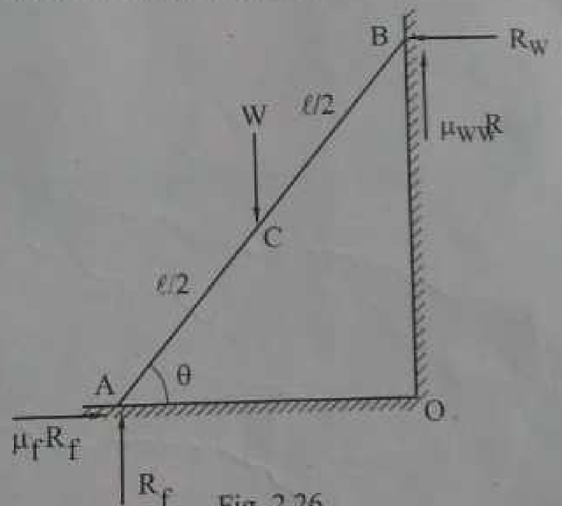


Fig. 2.26

will be towards the wall. For equilibrium of ladder the algebraic sum of vertical forces a[...] algebraic sum of horizontal forces must be zero. Also the sum of moments of all the forc[...] about any point must be zero

For limiting equilibrium,

For $\Sigma F_H = 0$,

$$\mu_f R_f - R_W = 0 \text{ -----(i)}$$

For $\Sigma F_V = 0$,

$$R_f + \mu_W R_W - W = 0 \text{ -----(ii)}$$

For $\Sigma M = 0$, taking moments of all the forces about the lower end A,

$$W \times AC \times \cos\theta - \mu_W R_W \times OA - R_W \times OB = 0$$

$$W \times \frac{l}{2}\cos\theta - \mu_W R_W \times l\cos\theta - R_W \times l\sin\theta = 0 \text{------(iii)}$$

Using the above three equations the limiting inclination of ladder with the floor can be calculated.

Example 2.9.

A uniform ladder 5m long, weighing 250 N, is placed against a smooth vertical wall with its lower end 2m from the wall. The coefficient of friction between the ladder and floor is 0.25. Show that the ladder will remain in equilibrium in this position.

Solution.

Since the wall is smooth, the frictional force at wall is zero.

Consider the free-body diagram of ladder AB.

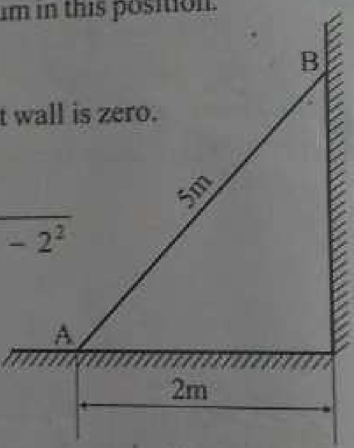


Fig. 2.27

$$BC = \sqrt{AB^2 - AC^2} = \sqrt{5^2 - 2^2}$$

$$= 4.58 \text{ m}$$

For $\Sigma F_V = 0$,

$$R_f - 250 = 0$$

$$R_f = 250 \text{ N}$$

The limiting frictional force is μR_N

$$= 0.25 \times 250$$

$$= 62.5 \text{ N}$$

Taking moments about B,

$$R_f \times AC - 250 \times 1 - F \times BC = 0$$

$$250 \times 2 - 250 \times 1 = F \times 4.58$$

$$F = 54.59 \text{ N}$$

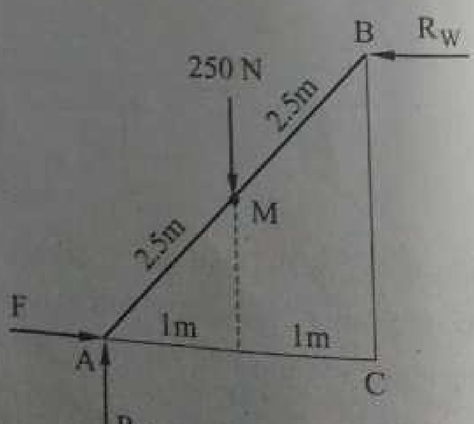


Fig. 2.28

Since the frictional force at A is less than the limiting frictional force of 62.5 N, the ladder will remain in equilibrium.

Example 2.10.

A uniform ladder 6 m long weighing 300 N, is resting against a wall with which it makes 30°. A man weighing 750 N climbs up the ladder. At what position along the ladder from the bottom end does the ladder slips? The coefficient of friction for both the wall and the ground with the ladder is 0.2.

Solution.

Consider the limiting equilibrium of the ladder.

For $\Sigma F_H = 0$

$$0.2\,R_f - R_W = 0$$

$$R_f = 5\,R_W$$

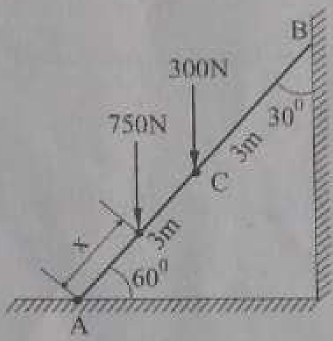


Fig. 2.29

For $\Sigma F_V = 0$,

$$R_f - 750 - 300 + 0.2\,R_W = 0$$

$$5\,R_W - 1050 + 0.2\,R_w = 0$$

$$R_W = \frac{1050}{5.2} = 201.92\text{ N}$$

For $\Sigma M = 0$, taking moments about A,

$$750 \times x\cos 60 + 300 \times 3\cos 60 - 0.2\,R_W \times 6\cos 60 - R_W \times 6\sin 60 = 0$$

$$375\,x + 450 - 0.6\,R_W - 6\frac{\sqrt{3}}{2}R_W = 0$$

$$375\,x + 450 = 201.92\,(0.6 + 3\sqrt{3})$$

$$x = 1.92\text{ m.}$$

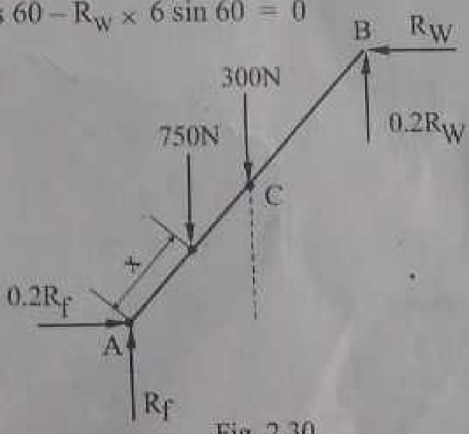


Fig. 2.30

Example 2.11.

A uniform ladder 3m long weighs 200N. It is placed against a vertical wall with which it makes an angle of 30°. The coefficient of friction between the wall and the ladder is 0.25 and that between the floor and ladder is 0.35. The ladder in addition to its own weight has to support a load of 1000N at its top end. Find, (i) the horizontal force P to be applied to the ladder at the floor level to prevent slipping.

(ii) If the force P is not applied, what should be the minimum inclination of the ladder with the horizontal so that there is no slipping of it with the load at its top end.

Solution.

Inclination of ladder with horizontal = 90 – 30 = 60^0

$l = 3$ m, $W = 200$ N, $\mu_w = 0.25$, $\mu_f = 0.35$

External load at the top = 1000 N

Case (i)

Consider the limiting equilibrium of the ladder

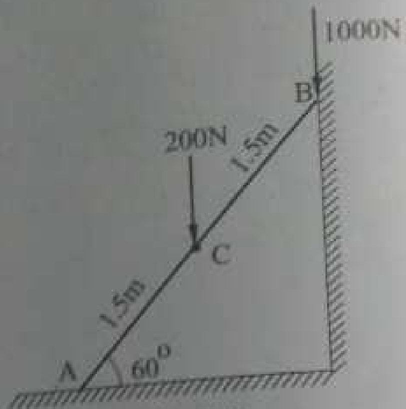


Fig. 2.31

For $\Sigma F_V = 0$,

$$R_f + 0.25\, R_W - 200 - 1000 = 0$$

$$R_f + 0.25\, R_W = 1200 \text{ ----(i)}$$

For $\Sigma F_H = 0$

$$P + 0.35\, R_f - R_W = 0$$

$$P = R_W - 0.35\, R_f \text{ -----(ii)}$$

For $\Sigma M = 0$, taking moments about A,

$$200 \times 1.5 \cos 60 + 1000 \times 3 \cos 60 - 0.25\, R_W\, 3 \cos 60 - R_W \times 3 \sin 60 = 0$$

$$150 + 1500 = R_W\ (0.25 \times 3 \cos 60 + 3 \sin 60)$$

$$R_W = 554.98 \text{ N}$$

From eqn (i), $R_f + 0.25\, R_W = 1200$

$$R_f = 1200 - 0.25 \times 554.98$$

$$= 1061.26 \text{ N}$$

From eqn (ii) $\quad P = R_W - 0.35\, R_f$

$$= 554.98 - 0.35 \times 1061.26$$

$$= 183.54 \text{ N}$$

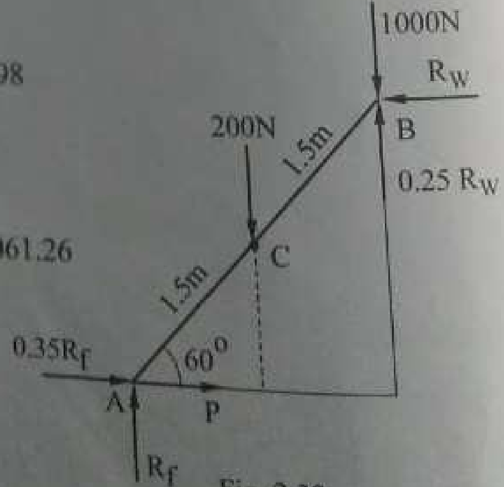


Fig. 2.32

Case (ii) Let the required inclination of ladder with horizontal be θ

For $\Sigma F_V = 0$,

$$R_f - 200 - 1000 + 0.25\, R_W = 0$$

$$R_f + 0.25\, R_W = 1200 \text{ N} \text{ ----(ii)}$$

For $\Sigma F_H = 0$

$$0.35\, R_f - R_W = 0$$

$$R_f = \frac{R_W}{0.35} = 2.86\, R_W$$

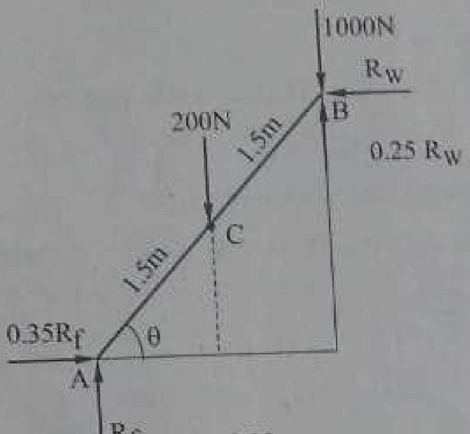


Fig. 2.33

From eqn (i)

$$R_f + 0.25\, R_W = 1200$$

$$2.86\, R_W + 0.25\, R_W = 1200$$

$$R_W = 385.85 \text{ N}$$

For $\Sigma M = 0$, taking moments about A,

$$200 \times 1.5 \cos\theta + 1000 \times 3\cos\theta - 0.25\, R_W \times 3\cos\theta - R_W \times 3\sin\theta = 0$$

$$\cos\theta\,(300 + 3000 - 0.25 \times 3 \times 385.85) = 3 \times 385.85 \sin\theta$$

$$\tan\theta = 2.60$$

$$\theta = 68.97^\circ$$

Example 2.12. [KTU May 2019]

A ladder AB 3m long weighs 180N, is placed against a wall with A at the floor level and B on the wall. The ladder is inclined at 60^0 with the floor . The coefficient of friction between the wall and the ladder is 0.25 and that between the floor and the ladder is 0.35. In additior to the self weight of the ladder, it has to support a man weighing 900N, at the top B. T prevent slipping a force is applied horizontally at the level of floor. Find the minimum horizontal force in this condition.

Solution.

Inclination of ladder with horizontal = 60^0

$l = 3m$, $W = 180N$, $\mu_W = 0.25$, $\mu_f = 0.35$

External load at the top = 900N

Consider the limiting equilibrium of the ladder

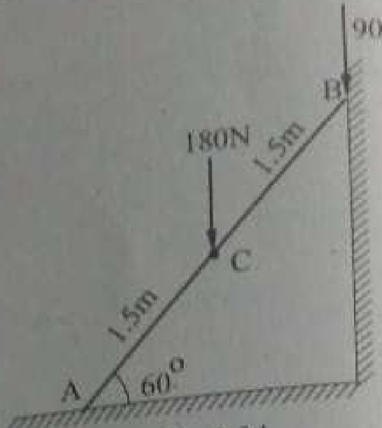


Fig. 2.34

For $\sum F_V = 0$,

$$R_f + 0.25\, R_W - 180 - 900 = 0$$

$$R_f + 0.25\, R_W = 1080 \text{ ----(i)}$$

For $\sum F_H = 0$

$$P + 0.35\, R_f - R_W = 0$$

$$P = R_W - 0.35\, R_f \text{ -----(ii)}$$

For $\sum M = 0$, taking moments about A,

$$180 \times 1.5 \cos 60 + 900 \times 3 \cos 60 - 0.25\, R_W 3 \cos 60 - R_W \times 3 \sin 60 = 0$$

$$135 + 1350 = R_W\ (0.25 \times 3 \cos 60 + 3 \sin 60)$$

$$R_W = 499.5N$$

From eqn (i), $R_f + 0.25\, R_W = 1080$

$$R_f = 1080 - 0.25 \times 499.5$$

$$= 955N$$

From eqn (ii) $P = R_W - 0.35\, R_f$

$$= 499.5 - 0.35 \times 955$$

$$= 165.5N$$

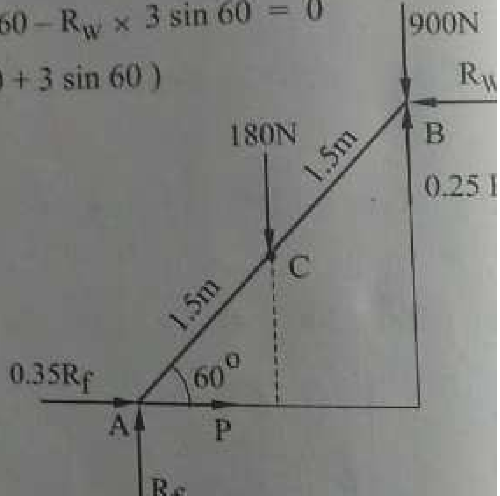


Fig. 2.35

Example 2.13. [KTU June 2016, July 2016, Jan 2017]

A uniform ladder of 4m length rests against a wall which it makes an angle 45° as sh in Fig. 2.36. The coefficient of friction between the ladder and the wall is 0.4 and between the ladder and the floor is 0.5. If a man whose weight is on half of that of la ascends it, how high will he when the ladder slips?

Solution.

Consider the limiting equilibrium of the ladder.

For $\Sigma F_H = 0$

$$0.5 R_f - R_W = 0$$

$$R_f = 2 R_W$$

For $\Sigma F_V = 0$,

$$R_f - W - \frac{W}{2} + 0.4 R_W = 0$$

$$2 R_W - 1.5W + 0.4 R_w = 0$$

$$2.4R_W = 1.5W$$

$$R_W = \frac{1.5W}{2.4} = 0.625\ W$$

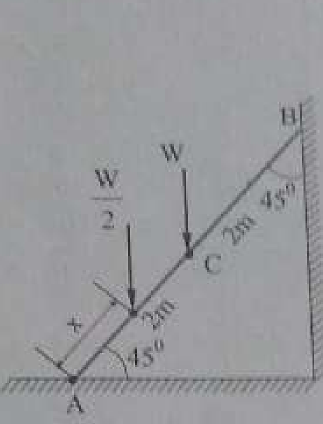


Fig. 2.36

For $\Sigma M = 0$, taking moments about A,

$$\frac{W}{2} x \cos 45 + W \times 2 \cos 45 - 0.4 R_W \times 4 \cos 45 - R_W \times 4 \sin 45 = 0$$

$$\frac{W}{2} x \cos 45 + W \times 2 \cos 45 - 0.4 \times 0.625\ W \times 4 \cos 45 - 0.625\ W \times 4 \sin 45 = 0$$

$$\frac{x}{2} + 2 - 0.4 \times 0.625 \times 4 - 0.625 \times 4 = 0$$

$$\frac{x}{2} = 1.5$$

$$x = 3m$$

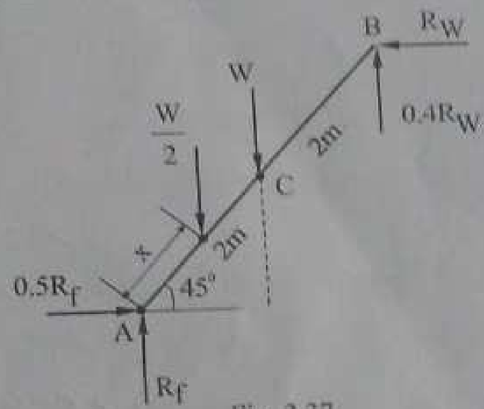


Fig. 2.37

Example 2.14 [KTU May 2017]

A ladder 5m long and weighing 260N, is placed against a vertical wall at an inclination o 30° with the wall. A man weighing 780N climbs the ladder. When he is at a distance of 1.64r along the ladder from the lower end, the ladder slips. What is the coefficient of frictio assuming it to be the same for all contact surfaces?

Solution.

Consider the limiting equilibrium of the ladder.

For $\Sigma F_H = 0$, $\mu R_f - R_W = 0$

$$R_f = \frac{R_w}{\mu} \quad \text{.................. (i)}$$

For $\Sigma F_V = 0,$

$$R_f - 780 - 260 + \mu R_W = 0$$

$$R_f + \mu R_w = 10.40$$

$$\frac{R_w}{\mu} + \mu R_w = 1040$$

$$R_w[\frac{1}{\mu} + \mu] = 1040$$

$$R_w = 1040\,[\frac{\mu}{1+\mu^2}]$$

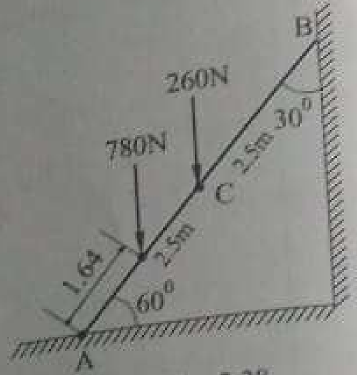


Fig. 2.38

For $\Sigma M = 0$, taking moments about A,

$$780 \times 1.64 \cos 60 + 260 \times 2.5 \cos 60 - \mu R_W \times 5 \cos 60 - R_W \times 5 \sin 60 = 0$$

$$964.6 = R_W[2.5\mu + 4.33]$$

$$= 1040\,[\frac{\mu}{1+\mu^2}]\,[2.5\mu + 4.33]$$

$$0.93 = \frac{2.5\mu^2 + 4.33\mu}{1+\mu^2}$$

$$0.93 + 0.93\mu^2 = 2.5\mu^2 + 4.33\mu$$

$$1.57\mu^2 + 4.33\mu - 1.57 = 0$$

$$\mu = 0.2$$

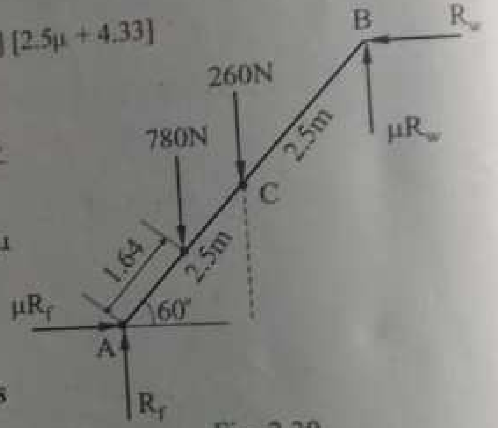


Fig. 2.39

2.7. Analysis of friction in connected bodies

Example 2.15.

Block A in Fig 2.40 weights 200 N and block B weights 300 N. Find the force P required to move block B. Assume the coefficient of friction for all surface as 0.3.

Solution.

Consider the upper block A, let T be the tension in the string.

For $\Sigma F_V = 0$

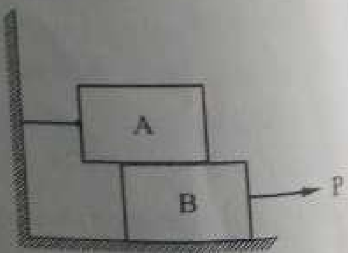


Fig. 2.40

$R_{N1} - W = 0$

$R_{N1} = W = 200\ N$

For $\Sigma F_H = 0$

$\mu R_{N1} - T = 0$

$0.3 \times 200 = T$

$T = 60\ N$

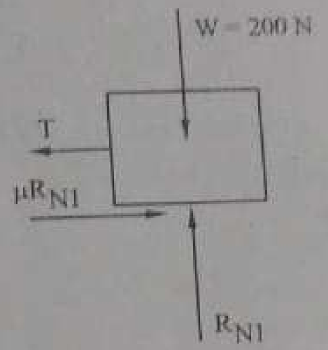


Fig. 2.41

Consider the lower block B.

Since $\Sigma F_V = 0$

$R_{N2} - W - R_{N1} = 0$

$R_{N2} = W + R_{N1}$

$= 300 + 200 = 500\ N$

For $\Sigma F_H = 0$

$P - \mu R_{N2} - \mu R_{N1} = 0$

$P = \mu R_{N2} + \mu R_{N1}$

$= 0.3 \times 500 + 0.3 \times 200$

$= 210\ N$

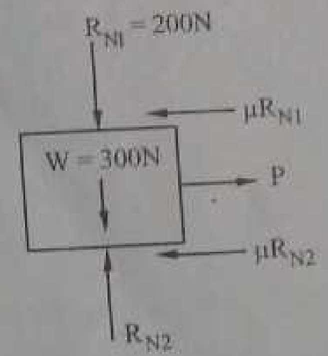


Fig. 2.42

Example 2.16.

What should be the value of angle θ for the motion of the block B weighing 90 N to impend down the plane. The coefficient of friction for all surfaces of contact is $\frac{1}{3}$. Block A weighs 30 N. The block A is held in position as shown in Fig. 2.43

Solution.

Consider the free - body diagram of block A.

Resolving the forces along the plane,

$T - \mu R_{N1} - W_A \sin\theta = 0$

$$T = 30 \sin\theta + \frac{1}{3} \times R_{N1}$$

Resolving the forces perpendicular to the inclined plane,

$$R_{N1} - W_A \cos\theta = 0$$

$$R_{N1} = 30 \cos\theta$$

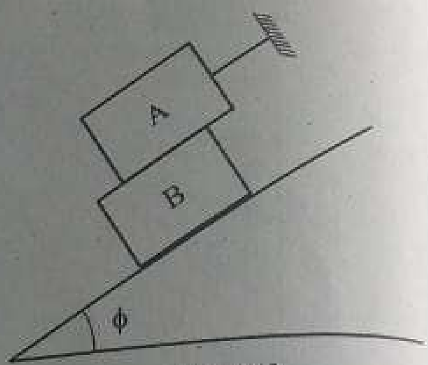


Fig. 2.43

Consider the free - body diagram of block B.

Resolving the forces perpendicular to the inclined plane,

$$R_{N2} - R_{N1} - W_B \cos\theta = 0$$

$$R_{N2} - 30 \cos\theta - 90 \cos\theta = 0$$

$$R_{N2} = 120 \cos\theta$$

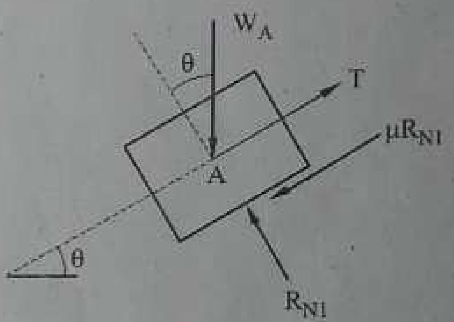


Fig. 2.44

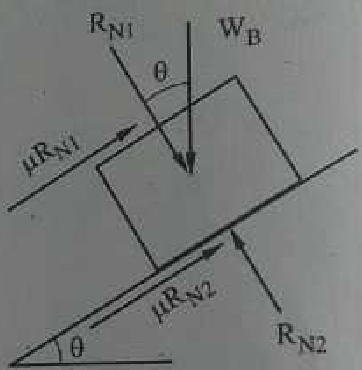


Fig. 2.45

Resolving the forces along the plane,

$$\mu R_{N2} + \mu R_{N1} - W_B \sin\theta = 0$$

$$\frac{1}{3}(120\cos\theta + 30\cos\theta) = 90\sin\theta$$

$$150 \cos\theta = 270 \sin\theta$$

$$\tan\theta = \frac{150}{270} = 0.555$$

$$\theta = 29.03^0$$

Example 2.17

Two blocks A and B of weights 500 N and 1000 N are placed on an inclined plane. The blocks are connected by a string parallel to the inclined plane. The coefficient of friction between the inclined plane and block A is 0.15 and that for the block B is 0.4. Find the inclination of plane when the motion is about to take place. Also calculate the tension in the string. The block A is below the block B as shown in Fig. 2.46.

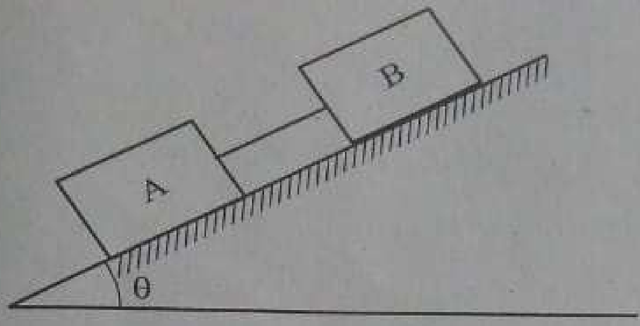


Fig. 2.46

Solution.

Let T be the tension in the string. Since the two bodies are in equilibrium under the action of forces, resolving the forces acting on block A along the plane,

$T + 0.15\,R_{NA} - 500 \sin\theta = 0$ ----(i)

Resolving the forces normal to the plane,

$R_{NA} - 500 \cos\theta = 0$

$R_{NA} = 500 \cos\theta,$

Substituting this value of R_{NA} in eqn (i)

$T + 0.15 \times 500 \cos\theta - 500 \sin\theta = 0$

$T = 500 \sin\theta - 75 \cos\theta$

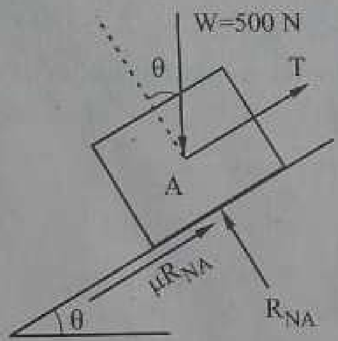


Fig. 2.47

Considering the equilibrium of block B and resolving the forces along the plane

$0.4\,R_{NB} - T - 1000 \sin\theta = 0$

$0.4\,R_{NB} - (500 \sin\theta - 75 \cos\theta) - 1000 \sin\theta = 0$

$0.4\,R_{NB} = 1500 \sin\theta - 75 \cos\theta$

Resolving the forces normal to the plane,

$R_{NB} - 1000 \cos\theta = 0$

$R_{NB} = 1000 \cos\theta$

$0.4\,(1000 \cos\theta) = 1500 \sin\theta - 75 \cos\theta$

$400 \cos\theta = 1500 \sin\theta - 75 \cos\theta$

$475 \cos\theta = 1500 \sin\theta$

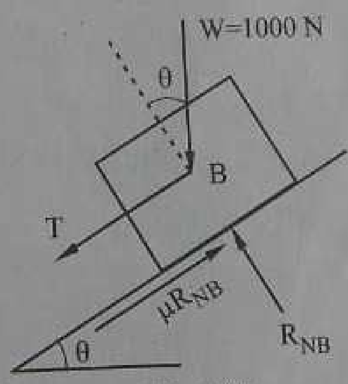


Fig. 2.48

$$\tan\theta = \frac{475}{1500} = 0.317$$

$$\theta = 17.59^0$$

$$T = 500\sin\theta - 75\cos\theta$$

$$= 500\sin 17.59 - 75\cos 17.59$$

$$= 79.61\ N$$

Example 2.18.

Two identical blocks, A and B, of weight W are supported by a rigid bar inclined 45° with horizontal as shown in Fig. 2.49. If both the blocks are in limiting equilibrium, find the coefficient of friction, assuming it to be the same at the floor and the wall.

Solution.

Since the bar is in compression, the compressive force is directed towards the ends of the bar as shown in the free - body diagram of blocks A and B.

Let the compressive force be S.

Consider the equilibrium of block A.

Resolving the forces horizontally,

$$R_{NA} - S\cos 45 = 0$$

$$R_{NA} = 0.707\ S$$

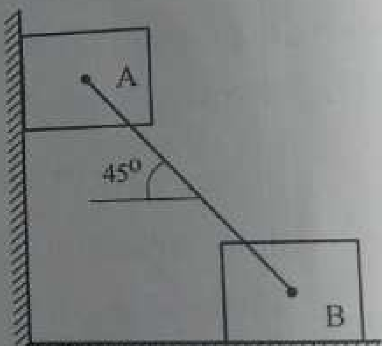


Fig. 2.49

Resolving the forces vertically,

$$\mu R_{NA} + S\sin 45 - W = 0$$

$$\mu \times 0.707\ S + 0.707\ S = W$$

$$0.707\ S\,(1 + \mu) = W \text{ -----(i)}$$

Consider the equilibrium of block B.

Resolving the forces vertically,

$$R_{NB} - W - S\sin 45 = 0$$

$$R_{NB} = W + 0.707\ S$$

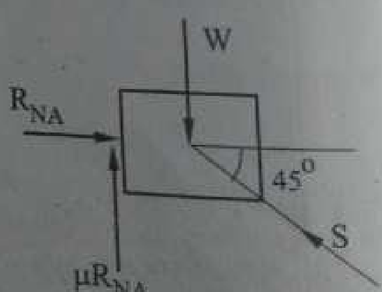


Fig. 2.50

Resolving the forces horizontally,

$$S\cos 45 - \mu R_{NB} = 0$$

$S \cos 45 - \mu (W + 0.707\ S) = 0$

$0.707\ S - \mu \times 0.707\ S = \mu W$

$0.707\ S\ (1 - \mu) = \mu W$ ----(ii)

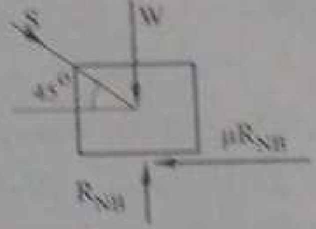


Fig. 2.51

From eqns (i) and (ii)

$$\frac{0.707\ S(1-\mu)}{0.707\ S(1+\mu)} = \frac{\mu W}{W}$$

$$1 - \mu = \mu\ (1 + \mu)$$

$$1 - \mu = \mu + \mu^2$$

$$\mu^2 + 2\ \mu - 1 = 0$$

$$\mu = \frac{-2 + \sqrt{4+4}}{2} = \frac{-2 + 2\sqrt{2}}{2}$$

$$= 0.414$$

2.8. Parallel coplanar forces.

The forces whose lines of action are parallel to each other are called parallel forces. Parallel forces may be like parallel forces or unlike parallel forces. Two parallel forces are said to be like when they act in the same direction. Two parallel forces are said to be unlike when they act in opposite directions.

(a) Like parallel forces. (b) Unlike parallel forces

Fig. 2.52

Example 2.19.

Resolve the force of 300 N shown in Fig. 2.53, into two parallel components.

(i) at B and C and (ii) at C and D

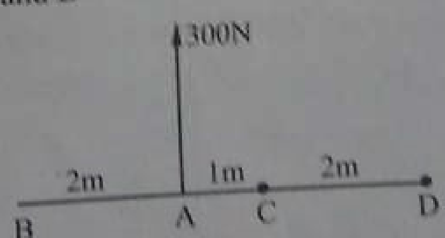


Fig. 2.53

Solution.

Let P and Q be the components.

$$P + Q = 300$$

Case (i) When the components are at B and C.

Moment of the given force about B is $300 \times 2 = 600$Nm. The magnitude of P and Q should be such that the sum of their moments about B should be 600Nm.

$$P \times 0 + Q \times 3 = 600$$
$$Q = 200 \text{ N}$$
$$P + Q = 300$$
$$P + 200 = 300$$
$$P = 100 \text{ N}$$

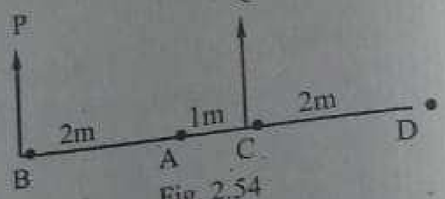


Fig. 2.54

Case (ii) When the components are at C and D.

$$P + Q = 300 \text{ N}$$

Moment of the given force about D is $300 \times 3 = 900$Nm. The magnitude of P and Q should be such that the sum of their moments about D should be 900 Nm.

$$P \times 2 + Q \times 0 = 900$$
$$P = 450 \text{ N}$$
$$P + Q = 300$$
$$450 + Q = 300$$
$$Q = 300 - 450$$
$$= -150 \text{ N}$$

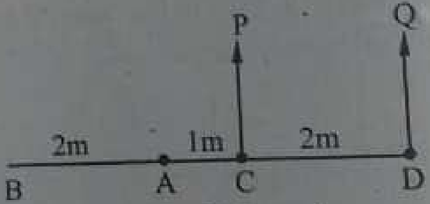


Fig. 2.55

Q is 150N downward.

Example 2.20.

Two unlike parallel forces are acting at a distance of 450 mm from each other. Th forces are equivalent to a single force of 900 N, which acts at a distance of 200 mm fror the greater of the two forces. Find the magnitude of the forces.

Solution.

Let P > Q

$$P - Q = 900 \text{ ----------(i)}$$

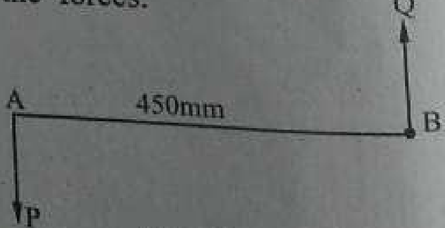


Fig. 2.56

Moment of P about B must be equal to moment of 900 N about B.

$$P \times 450 = 900 \times 650$$

$$P = 1300 \text{ N}$$

From eqn (i)

$$1300 - Q = 900$$

$$Q = 1300 - 900$$

$$Q = 400 \text{ N.}$$

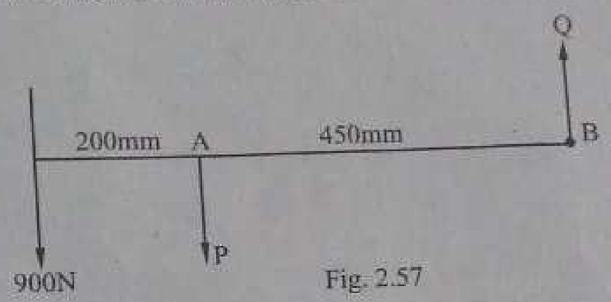


Fig. 2.57

2.9. Couple

The resultant of two unlike parallel forces of same magnitude is zero. Hence these forces cannot be replaced by another single force. Such two forces having the same magnitude, parallel line of action, and opposite sense are said to form a couple.The plane in which the forces act is called the plane of the couple. The distance between the line of action of forces is called arm of couple.

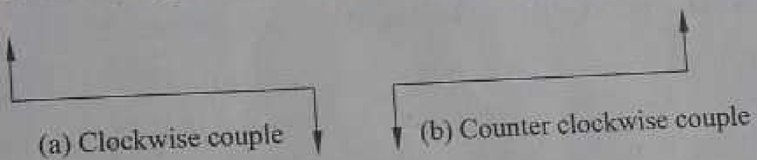


Fig. 2.58

Properties of force couple.

The two forces constituting a couple are equal in magnitude and opposite in direction. Therefore, the sum of forces of a couple is zero.

$$\sum F = 0$$

The moment of a couple about any point in the plane of couple is a constant and is independent of the position of moment centre.

Moment of couple about O,

$$M = F(d+x) - F x$$

$$= F d + F x - F x$$

$$= F d$$

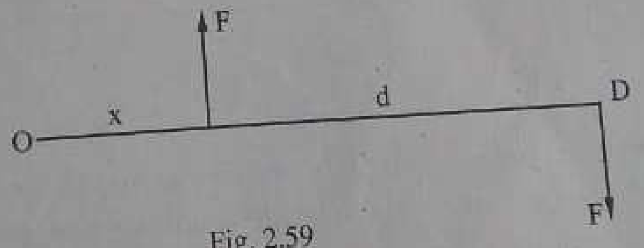


Fig. 2.59

Since the moment of couple, $F \times d$ is independent of x, it is a constant.

Another property of couple is that the action of a couple on a rigid body will not be changed if its arm is turned in the plane of couple through any angle about one of its ends, as shown in Fig. 2.60.

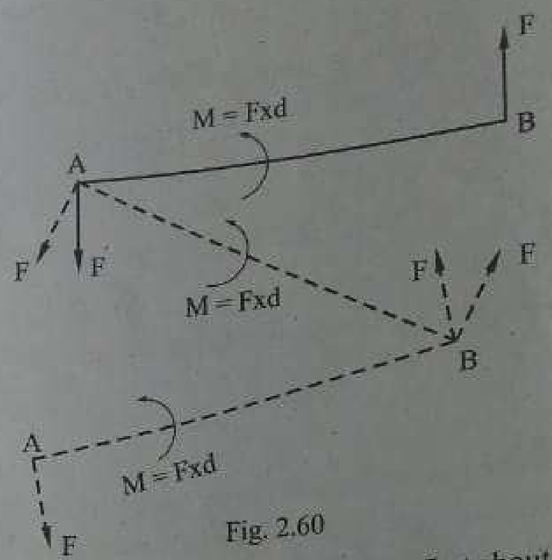


Fig. 2.60

By successive rotation of the arm of a couple in its plane, first about one end and then about the other, the couple can be put in any desired position in its plane. Thus we can transpose a couple in its plane without changing its action on a body.

The action of a couple on a body does not change if both the magnitudes of the forces and the arm of the couple are changed in such a way that the moment of the couple remains unchanged. That is, without changing the action on a body, a given couple can be replaced by another one with different forces with a different arm, provided the moments of the two couples are equal.

The couple consisting of magnitude of forces F and arm of couple d can be replaced by th couple of magnitude of force Q and arm of force d_1, if $F \times d = Q \times d_1$. Refer Fig. 2.61.

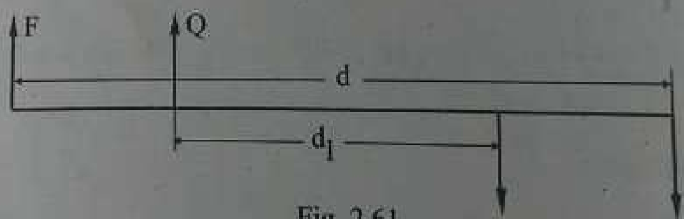


Fig. 2.61

Several couples in one plane can be replaced by a single couple acting in the same pla such that the moment of this single couple is equal to the algebric sum of the moments of t given couples. Refer Fig. 2.62.

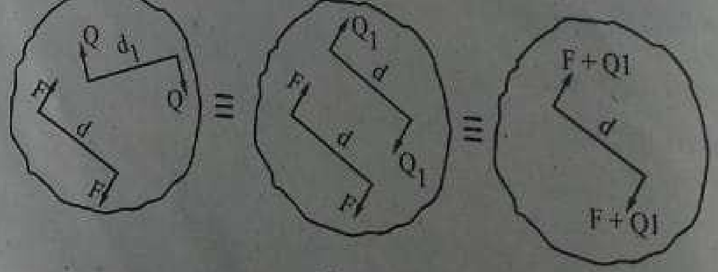


Fig. 2.62

$$F \times d + Q\,d_1 = F \times d + Q_1 \times d$$

$$= (F + Q_1)\,d \text{ where } Q_1 \text{ is given by}$$

$$Q_1 \times d = Q \times d_1$$

$$Q_1 = \frac{Q\,d_1}{d}$$

A given couple can be resolved into several component couples by choosing the component couples in such a manner that the algebric sum of their moments is equal to the moment of the given couple.

Example 2.21.

ABCD is a square whose side length is 2 m. Forces of magnitude 10,20,80 and 50 N act along AB, BC, CD and DA respectively. Forces of magnitude $50\sqrt{2}$ N and $20\sqrt{2}$ N act along the diagonal AC and DB respectively. Show that they are equivalent to a couple and calculate the moment of this couple.

Solution.

Resolving the forces horizontally,

$$\Sigma F_H = 10 - 80 + 50\sqrt{2} \times \cos 45 + 20\sqrt{2} \times \cos 45$$

$$= 0$$

$$\Sigma F_V = 20 - 50 + 50\sqrt{2}\cos 45 - 20\sqrt{2}\cos 45$$

$$= 20 - 50 + 50 - 20$$

$$= 0$$

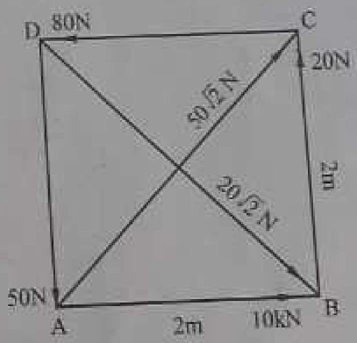


Fig. 2.63

Resultant force R = 0.

Taking moments of forces about A,

$$\Sigma M_A = (20\sqrt{2}\cos 45) \times 2 - 80 \times 2 - 20 \times 2$$

$$= 40 - 160 - 40 = -160 \text{ N m}$$

$$= 160 \text{ N m c.c.w.}$$

Since the resultant force is zero and the moment is not equal to zero, the system is equivalent to a couple of moment 160 Nm c.c.w.

Resolution of a given force into a force acting at a given point and a couple.

A given force F applied to a point A can be moved along its line of action but cannot be moved to any point away form the line of action without modifying the action of F on the rigi body. Consider a point B which is at a distance 'd' from the line of action of force F. Two oppositely directed collinear forces each of magnitude F and line of action parallel to the line of action of the given force F at A are introduced at B. The force F at A and oppositely directed force F at B constitute a couple of moment $M = F \times d$.

Thus a force F acting at A can be resolved into a force acting at B together with a couple of magnitude $F \times d$ as shown in Fig. 2.64.

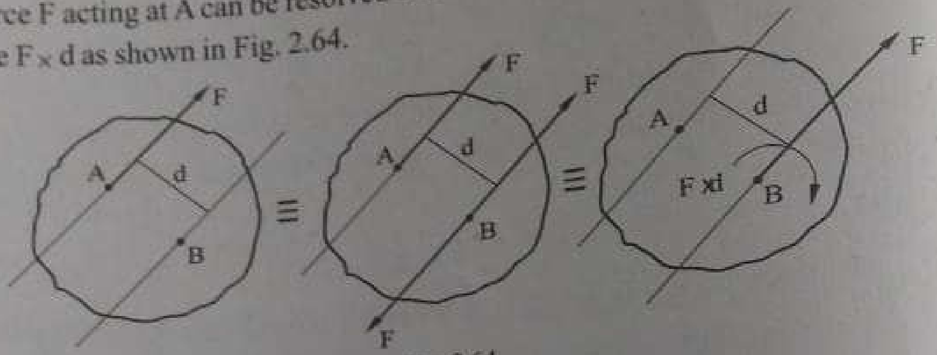


Fig. 2.64

Example 2.22

Replace the force acting at A by a force and couple at (i) B and (ii) at C.

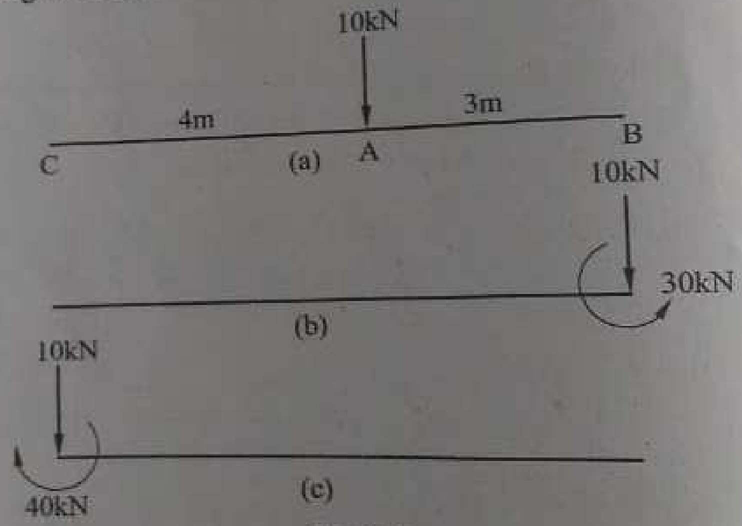


Fig. 2.65

Solution.

When the force acts at A, $\sum F_V = -10$ kN,

$\sum F_H = 0$ and $\sum M_B = -10 \times 3 = -30$ kNm $= 30$ kNm c.c.w

When the force 10 kN acts at B, as shown in Fig. 2.65(b), $\sum F_V = -10$ kN, $\sum F_H =$ and $\sum M_B = 0$. Therefore a counter clockwise moment of magnitude 30kNm should applied at B.

(ii) When the force acts at A, $\sum F_V = -10$ kN, $\sum F_H = 0$ and

$$\sum M_C = 10 \times 4 = 40 \text{ kNm c.w.}$$

When the force acts at C, as shown in Fig. 2.65(c), $\Sigma F_V = -10$ kN, $\Sigma F_H = 0$ and

$\Sigma M_C = 0$

Therefore a clockwise moment of magnitude 40 kNm should be applied at C.

2.10. Resultant of parallel forces.

Since all the forces are parallel, resultant will be the algebraic sum of the forces. Consider four parallel forces as shown in Fig. 2.66. The resultant $R = F_1 + F_2 - F_3 + F_4$.

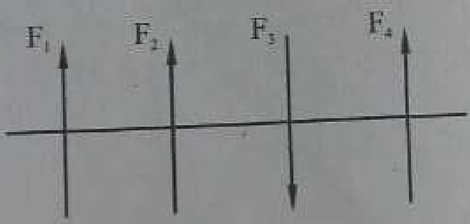


Fig. 2.66

Example 2.23.

Calculate the resultant of the force system of parallel forces as shown in Fig. 2.67.

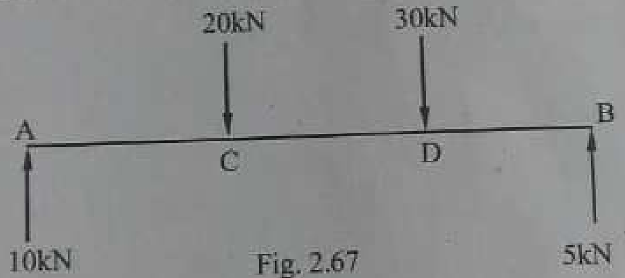


Fig. 2.67

Solution

Resultant R = 10 – 20 – 30 + 5

= – 35kN

= 35kN, downwards

2.11. Center of parallel forces

Resultant of a number of parallel forces is the single force by which the parallel forces can be replaced. It is the algebric sum of the parallel forces. The point of application of this resultant force is the centre of parallel forces. To locate this point the principle of moments can be applied. The sum of moments of all the forces in a plane about any point in the plane is equal to the moment of their resultant about the same point.

Example 2.24

Determine the resultant of the system of forces shown in Fig. 2.68.

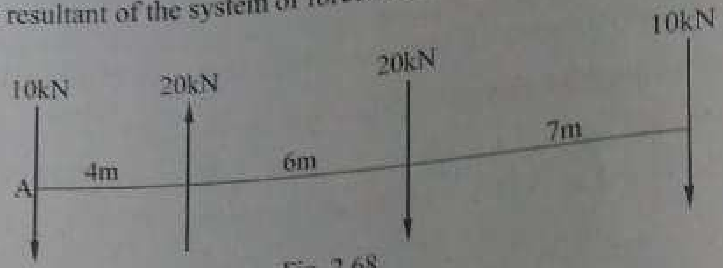


Fig. 2.68

Solution.

Since there is no horizontal or inclined force,

$$\Sigma F_H = 0$$

$$\Sigma F_V = -10 + 20 - 20 - 10 = -20 \text{ kN.}$$

$$\text{Resultant } R = \sqrt{(\Sigma F_H)^2 + (\Sigma F_V)^2}$$

$$\text{Resultant } R = \sqrt{(\Sigma F_H)^2 + (\Sigma F_V)^2}$$

$$= \sqrt{0 + (-20)^2}$$

$$= 20 \text{ kN}$$

Inclination of resultant with horizontal

$$\theta = \tan^{-1} \frac{\Sigma F_V}{\Sigma F_H} = \tan^{-1} \frac{20}{0}$$

$$= \tan^{-1} \infty = 90^0$$

Inclination of resultant with positive X axis, $\theta_R = 180 + \theta = 270^0$

Let the distance of line of action of this force from end A be x,

$$\text{then}, \Sigma M_A = R \times x$$

$$\Sigma M_A = 10 \times 0 - 20 \times 4 + 20 \times 10 + 10 \times 17$$

$$= -80 + 200 + 170 = 290 \text{ kNm C.W.}$$

Moment of R about A should be clockwise, for this R must be towards right of A.

Let it be at a distance x from A

$\Sigma M_A = Rx$

$x = \dfrac{\Sigma M_A}{R}$

$= \dfrac{290}{20} = 14.5 \text{ m}$

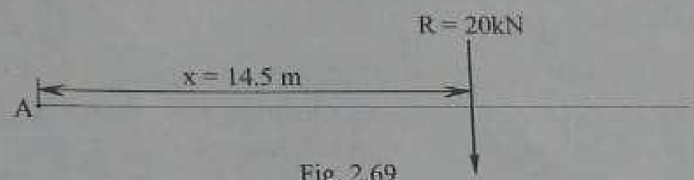


Fig. 2.69

Example 2.25 [KTU Aug 2016]

A rigid bar AB is acted upon by forces as shown in Fig. 2.70. Reduce the force system to (i) a single force (ii) force moment system at A, and (iii) force moment system at D.

Solution

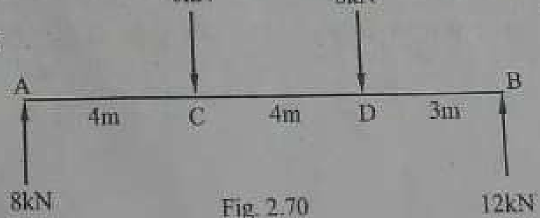


Fig. 2.70

$\Sigma F_V = 8 - 6 - 8 + 12 = 6 \text{ kN}$

Case (i)

$R = \sqrt{(\Sigma F_H)^2 + (\Sigma F_V)^2}$

$= \sqrt{0^2 + 6^2} = 6 \text{ kN}$

Let the resultant be at a distance x from A.

$\Sigma M_A = 6 \times 4 + 8 \times 8 - 12 \times 11 = -44 \text{ kN m}.$

$= 44 \text{ kNm . c.c.w}$

Moment of resultant about A = $6 \times x$

$6x = 44$

$x = \dfrac{44}{6} = 7.33 \text{ m}$

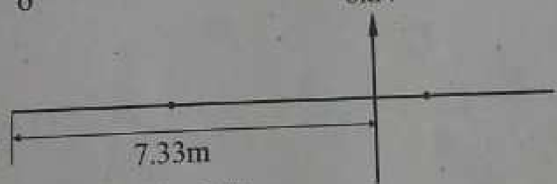


Fig. 2.71

Case (ii)

Sum of moments of forces about A is 44 kNm. The resultant force is 6 kN upwards. Therefore, the system can be reduced to a force moment system at A as shown in Fig. 2.72.

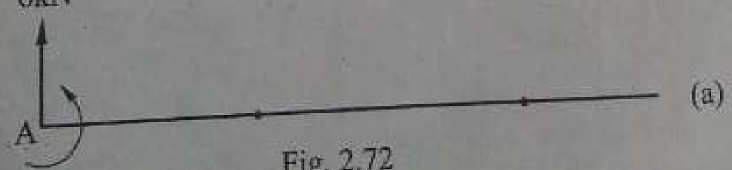
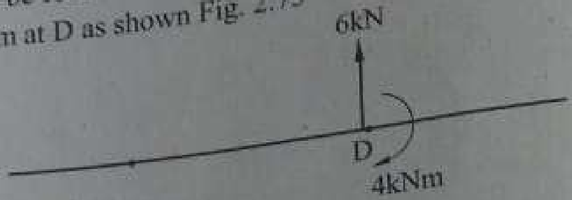


Fig. 2.72

Case (iii)

The sum of moments of forces about D is $8\times8-6\times4-12\times3=4$ kNm, c.w. Therefore, the given force system can be reduced to a single force of magnitude 6 kN along with a clockwise moment of 4 kNm at D as shown Fig. 2.73



(b)

Fig. 2.73

Example 2.26. [KTU July 2016]

A system of parallel forces is acting on a rigid bar as shown in Fig. 2.74. Reduce this system into (a) a single force (b) a single force and couple at A.

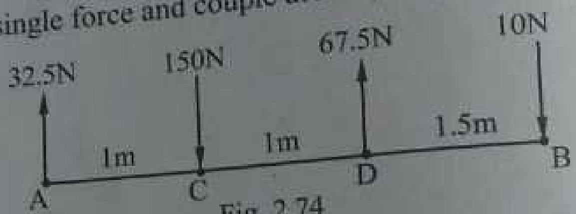


Fig. 2.74

Solution.

$$\Sigma F_V = 32.5-150+67.5-10=-60N$$

$$= 60N \text{ downwards}$$

Since $\Sigma F_H = 0$, resultant $R = \Sigma F_V = 60N$ downwards.

Case (i)

$$\Sigma M_A = 150\times1-67.5\times2+10\times3.5$$

$$= 50 \text{ Nm}$$

Fig. 2.75

$$\Sigma M_A = R\times x$$

$$50 = 60\times x$$

$$x = 0.833m$$

The given system can be reduced to a single downward force of magnitude 60N at a tance of 0.833m towards right of A.

Case (ii)

When the resultant is at a distance of 0.833m from A, the moment about A is 5 When this force acts at A the moment of this force is 0. Therefore a moment of should be added at A.

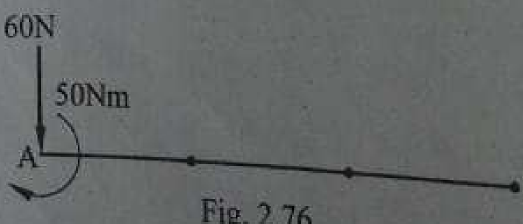


Fig. 2.76

2.12. Equilibrium of parallel forces.

For equilibrium of parallel forces, the resultant force and the resultant moment should be zero. The necessary and sufficient conditions for the equilibrium of parallel forces can be expressed analytically,

$$\sum F = 0 \text{ and } \sum M = 0.$$

Example 2.27.

A uniform beam AB of weight 100 kN and 6 m long had two bodies of weights 60 kN and 80 kN suspended from its two ends as shown in Fig. 2.77. At what point the beam should be supported that it may rest horizontally.

Fig. 2.77

Solution.

The position of support should be such that the sum of moments of all the forces about that point must be zero.

Let the required support be at a distance x towards left of B.

Taking moments of forces about the support,

$$80 \times x - 100(3 - x) - 60(6 - x) = 0$$

$$80x = 300 - 100x + 360 - 60x$$

$$240x = 660$$

$$x = \frac{660}{240} = 2.75 \text{ m.}$$

The support should be 2.75 m towards left of point B.

Example 2.28

A beam AB is hinged at A and supported at B by a vertical cord which passes over a frictionless pulley at C and carries at its end a load P as shown in Fig 2.78. Determine the distance 'x' from 'A' at which a load Q must be placed on the beam if it is to remain in equilibrium in a horizontal position. Neglect the weight of beam. If proportion of distance is x/l is 1: 6 and P = 10 N, find the load Q.

Solution

Consider the equilibrium of the load P. Let the tension in the string be T.

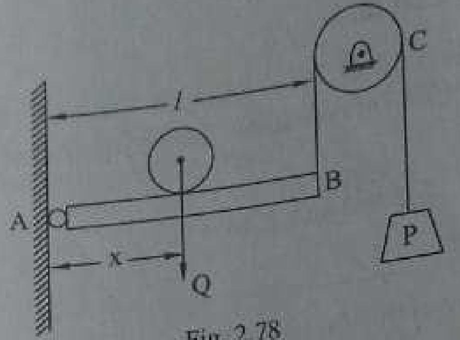


Fig. 2.78

For $\Sigma F_y = 0$

$$T - P = 0$$

$$T = P$$

Since the pulley is frictionless the tension in the cord, supporting the beam AB , T = P

Consider the equilibrium of the beam AB

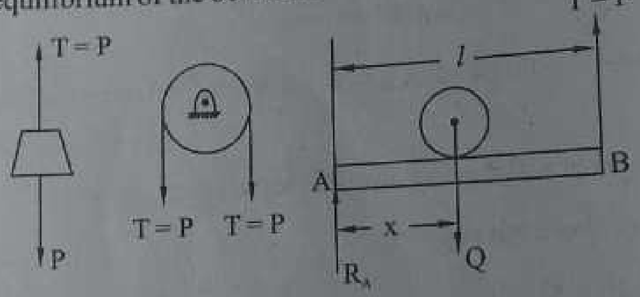


Fig. 2.79

For $\Sigma M = 0$, taking moments about A,

$$Q \times x - P \times l = 0$$

$$x = \frac{Pl}{Q}$$

$$Q = \frac{P \times l}{x}$$

$$Q = 10 \times \frac{6}{1} = 60N$$

2.13. Simple beams subjected to concentrated vertical loads

A structural element which has one dimension considerably larger than the other dimensions and which offers resistance to bending due to applied loads is known as be

To apply external load on the beam, it should be supported. When the end of a beam is kept simply on a smooth flat surface, the support is called simple support. Due to the applied loads, reactions are developed at the supports and the system of forces consisting of applied loads and reactions keep the beam in equilibrium. The nature of reaction depends upon the type of supports and type of loads. When a beam is subjected to vertical concentrated loads alone, the direction of reaction will be perpendicular to the supporting surface.

Example 2.29 [KTU July 2016]

A beam 6 m long is loaded as shown in Fig 2.80. Calculate the reactions at A and B.

Solution.

Consider the free -body diagram of beam as shown in Fig. 2.81.

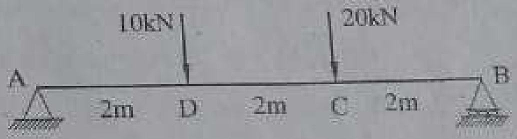


Fig. 2.80

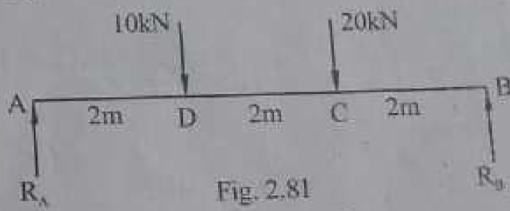


Fig. 2.81

For $\Sigma F_V = 0$,

$$R_A - 10 - 20 + R_B = 0$$

$$R_A + R_B = 30 \text{ ---------(i)}$$

For $\Sigma M = 0$, taking moments about A,

$$10 \times 2 + 20 \times 4 - R_B \times 6 = 0.$$

$$R_B = \frac{100}{6} = 16.67 \text{ kN}$$

$$R_A + R_B = 30$$

$$R_A + 16.67 = 30$$

$$R_A = 30 - 16.67$$

$$= 13.33 \text{ kN}$$

Example 2.30

A beam 6 m long is loaded as shown in Fig. 2.82. Calculate the reactions at supports A and B.

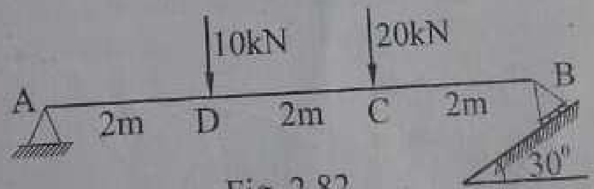


Fig. 2.82

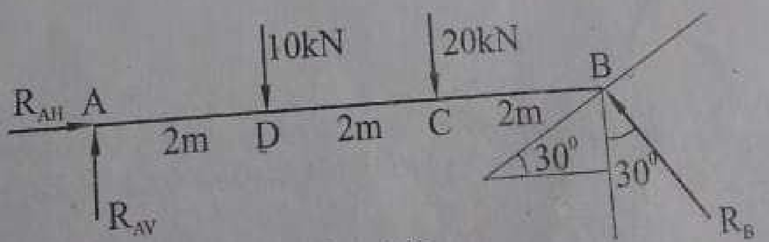


Fig. 2.83

Solution

Consider the freebody diagram of beam shown in Fig 2.83.

For $\Sigma F_H = 0$,

$R_{AH} - R_B \sin 30 = 0$

$R_{AH} = R_B \sin 30$ --------(i)

For $\Sigma F_V = 0$,

$R_{AV} - 10 - 20 + R_B \cos 30 = 0$

$R_{AV} + R_B \cos 30 = 30$ --------(ii)

For $\Sigma M = 0$, taking moments about A.

$10 \times 2 + 20 \times 4 - (R_B \cos 30) \times 6 = 0$

$20 + 80 = 6 R_B \cos 30$

$$R_B = \frac{100}{6 \cos 30} = 19.25 \text{ kN}$$

From eqn (i),

$R_{AH} = R_B \sin 30$

$= 19.25 \times 0.5 = 9.625$ kN

From eqn (ii),

$R_{AV} + 19.25 \cos 30 = 30$

$R_{AV} = 30 - 19.25 \cos 30$

$= 13.33$ kN.

$$R_A = \sqrt{(R_{AH})^2 + (R_{AV})^2}$$

$$= \sqrt{9.625^2 + 13.33^2} = 16.44 \text{ kN}$$

Inclination of resultant with vertical,

$$\theta_A = \tan^{-1} \frac{R_{AH}}{R_{AV}}$$

$$= \tan^{-1} \left(\frac{9.625}{13.33}\right)$$

$$= 35.83^0$$

Example 2.31

A beam 6m long is loaded as shown in Fig 2.84. Calculate the reactions at A and B.

Solution.

Consider the freebody diagram of beam shown in Fig. 2.85.

For $\Sigma F_V = 0$

$$R_A - 10 - 20 + R_B = 0$$

$$R_A + R_B = 30 \quad \text{------(1)}$$

For $\Sigma M = 0$, taking moments about A,

$$10 \times 2 + 20 \times 4 + 40 - R_B \times 6 = 0$$

$$20 + 80 + 40 = 6 R_B$$

$$R_B = \frac{140}{6} = 23.33 \text{ kN}$$

$$R_A + R_B = 30$$

$$R_A = 30 - R_B = 30 - 23.33$$

$$= 6.67 \text{ kN.}$$

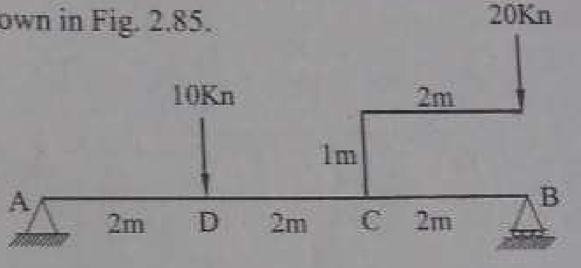


Fig. 2.84

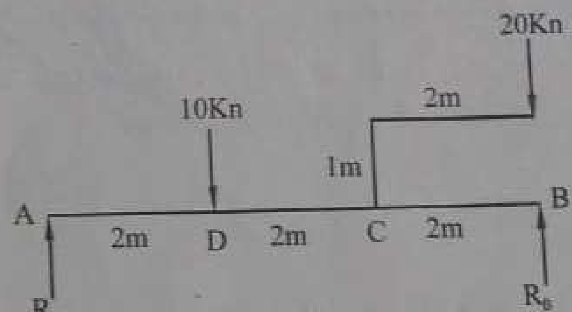


Fig. 2.85

2.14. General coplanar force system

Various forces acting in a plane may consist of concurrent forces, parallel forces and non - concurrent forces. Non - concurrent forces in a plane constitute a general system of coplanar forces.

2.15. Resultant of coplanar force system

Resultant of coplanar force system can be obtained by resolving the forces horizontally and vertically, or resolving along any two mutually perpendicular directions. Magnitude of resultant,

$$R = \sqrt{\Sigma F_H{}^2 + \Sigma Fv^2}$$

The inclination resultant with vertical is given by,

$$\tan\theta = \frac{\Sigma F_H}{\Sigma F_V}$$

The position of the resultant force can be located by applying principle of moments. The sum of moments of all the forces about any point is equal to the moment of the resultant about the same point.

Example 2.32

Three forces 200 N, 300 N and 400 N act along three sides of an equilateral triangle taken in order. Find the magnitude and line of action of the resultant force.

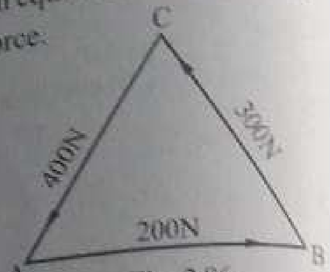


Fig. 2.86

Solution.

$$\Sigma F_H = 200 - 300 \cos 60 - 400 \cos 60$$

$$= 200 - 150 - 200 = -150 \text{ N}$$

$$\Sigma F_V = 300 \sin 60 - 400 \sin 60$$

$$= -86.6 \text{ N}$$

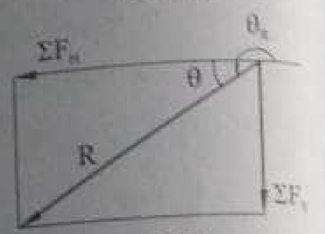


Fig. 2.87

$$\text{Resultant } R = \sqrt{(-150)^2 + (-86.6)^2}$$

$$= 173.2 \text{ N}$$

Inclination of resultant with horizontal

$$\theta = \tan^{-1}\left|\frac{\Sigma F_V}{\Sigma F_H}\right| = \tan^{-1}\frac{86.6}{150} = 30^0$$

Since both ΣF_H and ΣF_V are negative, the resultant is in the third quadrant.

$$\theta_R = 180 + \theta = 180 + 30 = 210^0$$

$$\Sigma M_A = (-300 \sin 60) \times AB = -259.81 \times AB$$

Let the line of action of resultant be at a distance x away from point A.

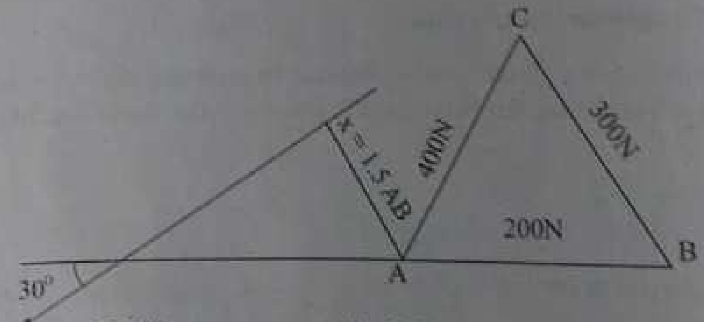


Fig. 2.88

$$\Sigma M_A = R \times x$$

$$x = \frac{\Sigma M_A}{R} = \frac{259.81 \times AB}{173.2} = 1.5 \text{ AB}$$

Example 2.33

Four forces equal to P, 2 P, 3 P and 4 P are acting along four sides of a square ABCD taken in order. Find the characteristics of the resultant.

Solution.

$$\Sigma F_V = 2P - 4P = -2P$$

$$\Sigma F_H = P - 3P = -2P$$

$$R = \sqrt{(\Sigma F_H)^2 + (\Sigma F_V)^2}$$

$$= \sqrt{(-2P)^2 + (-2P)^2}$$

$$= 2.828\ P$$

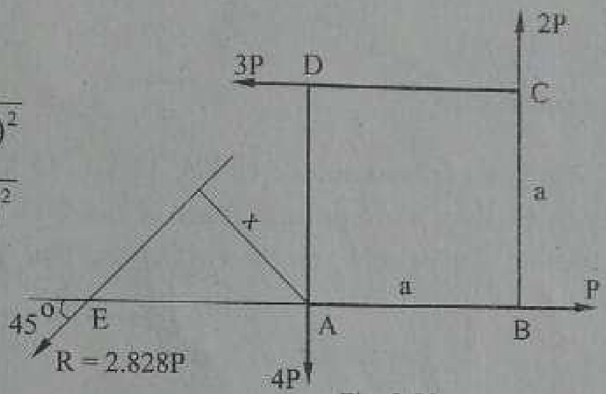


Fig. 2.89

Inclination of resultant with horizontal

$$\theta = \tan^{-1}\left|\frac{\Sigma F_V}{\Sigma F_H}\right|$$

$$= \tan^{-1}\left(\frac{2P}{2P}\right) = \tan^{-1} 1 = 45^0$$

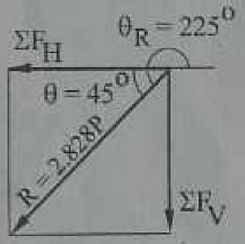


Fig. 2.90

Inclination of the resultant $\theta_R = 180 + \theta$

$$= 180 + 45$$

$$= 225^0$$

Sum of moments of the given forces about A,

$$\Sigma M_A = -2P \times a - 3P \times a$$

$$= -5P \times a$$

$$= 5Pa \text{ c.c.w}$$

Since the sum of moments of the given forces about A is counter clockwise, the moment of resultant R about A should also be counter clockwise. For this the position of resultant must be to the left of point A. Let the resultant be at a distance x from A, then moment of resultant about A is equal to R × x.

Therefore, $R \times x = 5P \times a$.

$$2.828\ P \times x = 5\ P \times a$$

$$x = 1.768\ a$$

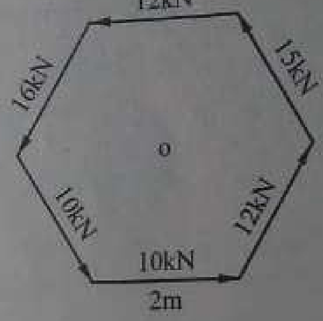
$$\frac{x}{AE} = \sin 45$$

$$AE = \frac{x}{\sin 45} = \frac{1.768\,a}{\sin 45}$$

$$= 2.5\ a$$

Example 2.34

Six forces of magnitude 10 kN, 12 kN, 15 kN, 12 kN, 16 kN and 10 kN are acting along the sides of a regular hexagon of side 2 m. in order . Find the resultant force and its direction. Find also the position of the resultant with respect to the centre of the hexagon.



Solution.

Fig. 2.91

$$\Sigma F_H = 10 + 12\cos 60 - 15\cos 60 - 12 - 16\cos 60 + 10\cos 60$$

$$= -\ 6.5\ \text{kN}$$

$$\Sigma F_V = 0 + 12\sin 60 + 15\sin 60 + 0 - 16\sin 60 - 10\sin 60$$

$$= 0.866\ \text{kN.}$$

$$\text{Resultant force } R = \sqrt{(\Sigma F_H)^2 + (\Sigma F_V)^2}$$

$$= \sqrt{(6.5)^2 + (0.866)^2}$$

$$= 6.56\ \text{kN.}$$

Inclination of resultant force with horizontal

$$\theta = \tan^{-1}\left|\frac{\Sigma F_V}{\Sigma F_H}\right|$$

$$= \tan^{-1}\left(\frac{0.866}{6.5}\right)$$

$$= 7.59^0$$

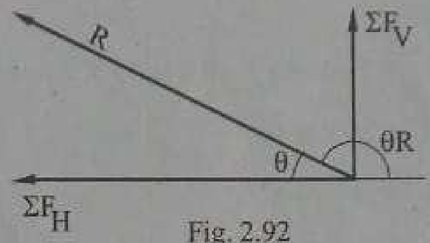


Fig. 2.92

Inclination of resultant $\theta_R = 180 - \theta$

$$= 180 - 7.59$$

$$= 172.41^0$$

Sum of moments of all the forces about the centre of hexagon,

$$\Sigma M = -(10 + 12 + 15 + 12 + 16 + 10) \times 2 \sin 60$$

$$= -75 \times 2 \sin 60$$

$$= 129.90 \text{ kNm c.c.w.}$$

Let the distance of resultant force from the centre be x.

Then, moment of resultant about the centre $= R \times x$.

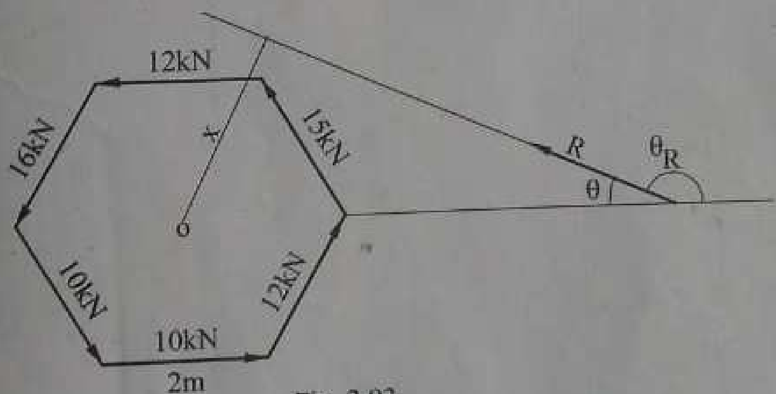


Fig. 2.93

Equating these two moments,

$$R \times x = 129.90$$

$$x = \frac{129.90}{6.56} = 19.80 \text{ m}$$

Example 2.35

Find the resultant of the forces shown in Fig 2.94.

Solution.

$\Sigma F_H = 5 \cos 45 + 10 \cos 45 + 5 \cos 45$

$= 20 \cos 45$

$= 14.14$ kN

$\Sigma F_V = 5 \sin 45 - 10 \sin 45 - 5 \sin 45$

$= -10 \sin 45$

$= -7.07$ kN

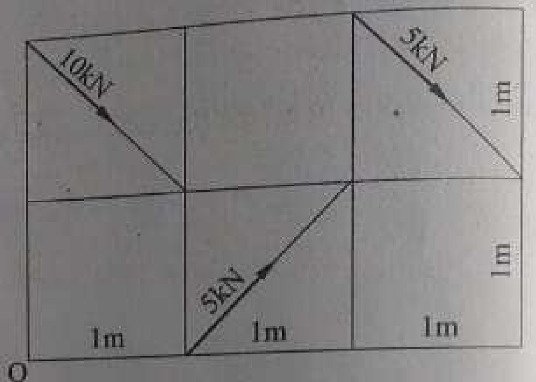


Fig. 2.94

$R = \sqrt{(\Sigma F_H)^2 + (\Sigma F_V)^2}$

$R = \sqrt{(14.14)^2 + (-7.07)^2}$

$= 15.81$ kN

Inclination of resultant with horizontal,

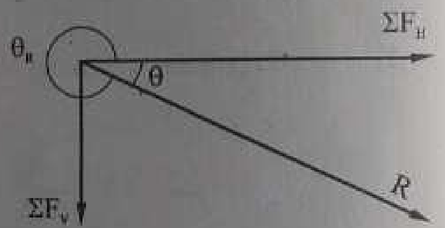


Fig. 2.95

$\theta = \tan^{-1} \left| \frac{\Sigma F_V}{\Sigma F_H} \right|$

$= \tan^{-1} \frac{7.07}{14.14}$

$= 26.57^0$

$\theta_R = 360 - \theta = 360 - 26.57$

$= 333.43^0$

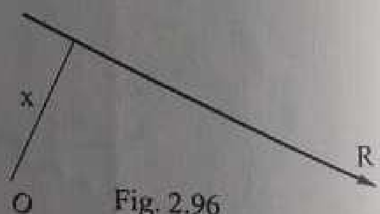


Fig. 2.96

Sum of moments of all the forces about O,

$\Sigma M_O = (10 \cos 45) \times 2 + (5 \cos 45) \times 2 + (5 \sin 45) \times 2 - (5 \sin 45) \times 1$

$= 24.75$ kNm

Moment of resultant about O = R × x = 24.75 k N m

$\therefore x = \frac{24.75}{R} = \frac{24.75}{15.81} = 1.57\text{m}$

Example 2.36

Determine the resultant of the force system shown in Fig. 2.97 with respect to point O. All squares are unit squares.

Solution.

$$\Sigma F_H = -\ 100\cos\theta_1 - 80\cos\theta_2 + 60\cos\theta_3 + 30\cos\theta_4$$

$$\Sigma F_V = -\ 100\sin\theta_1 + 80\sin\theta_2 + 60\sin\theta_3 - 30\sin\theta_4$$

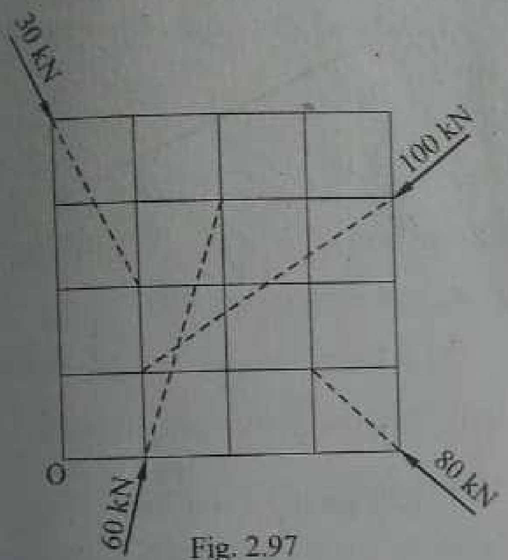


Fig. 2.97

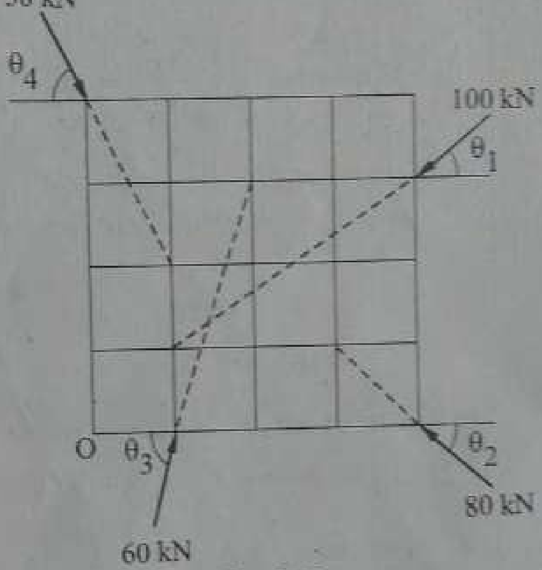


Fig. 2.98

$$\tan\theta_1 = \frac{2}{3};\ \theta_1 = 33.69^0$$

$$\tan\theta_2 = \frac{1}{1};\ \theta_2 = 45^0$$

$$\tan\theta_3 = \frac{3}{1};\ \theta_3 = 71.56^0$$

$$\tan\theta_4 = \frac{2}{1};\ \theta_4 = 63.43^0$$

$$\Sigma F_H = -\ 100\cos 33.69^0 - 80\cos 45^0 + 60\cos 71.56^0 + 30\cos 63.43^0$$

$$= -\ 107.38\ kN$$

$$\Sigma F_v = -100 \sin 33.69^\circ + 80 \sin 45^\circ + 60 \sin 71.56^\circ - 30 \sin 63.43^\circ$$

$$= 31.19 \text{ kN}$$

$$R = \sqrt{(\Sigma F_H)^2 + (\Sigma F_V)^2}$$

$$= \sqrt{(-107.38)^2 + (31.19)^2}$$

$$= 111.82 \text{ kN}$$

Since ΣF_H is negative and ΣF_v is positive, the resultant is in the second quadrant. Its inclination with the horizontal.

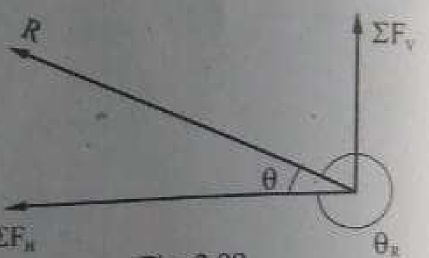


Fig. 2.99

$$\theta = \tan^{-1}\left|\frac{\Sigma F_V}{\Sigma F_H}\right| = \tan^{-1}\left|\frac{31.19}{107.38}\right|$$

$$= 16.20^\circ$$

$$\theta_R = 180 - \theta$$

$$= 180 - 16.20$$

$$= 163.80^\circ$$

Sum of moments of all the forces about O,

$$\Sigma M_0 = 30 \cos \theta_4 \times 4 - 60 \sin \theta_3 \times 1 - 80 \sin \theta_2 \times 4 + 100 \sin \theta_1 \times 4 - 100 \cos \theta_1 \times 3$$

$$= 30 \cos 63.43 \times 4 - 60 \sin 71.56 \times 1 - 80 \sin 45 \times 4 + 100 \sin 33.69 \times 4 - 100 \cos 33.69 \times 3.$$

$$= -257.25$$

$$= 257.25 \text{ kNm C.C.W.}$$

Moment of resultant about O = $R \times x$

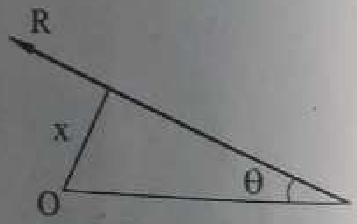


Fig. 2.100

$$R \times x = \Sigma M_0$$

$$x = \frac{257.25}{111.82} = 2.3 \text{ m}$$

Example 2.37

Find the resultant of a set of coplanar forces as shown in Fig. 2.101.

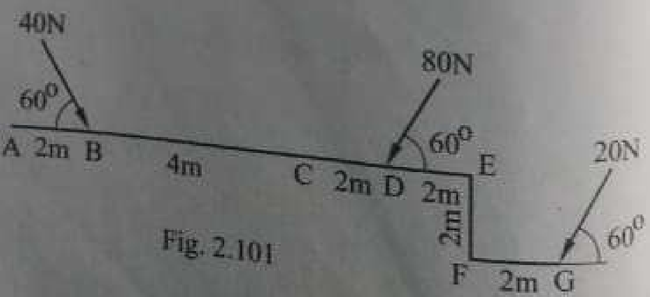


Fig. 2.101

Solution

Resolving the forces horizontally,

$$\Sigma F_H = 40 \cos 60 - 80 \cos 60 - 20 \cos 60$$
$$= 20 - 40 - 10 = -30 \text{ N}$$

Resolving the forces vertically,

$$\Sigma F_V = -40 \sin 60 - 80 \sin 60 - 20 \sin 60. = -121.24 \text{ N}$$

Resultant force $R = \sqrt{(\Sigma F_H)^2 + (\Sigma F_V)^2} = \sqrt{(-30^2) + (121.24)^2}$

$$= 124.90 \text{ N}$$

Inclination of resultant with horizontal,

$$\theta = \tan^{-1}\left|\frac{\Sigma F_V}{\Sigma F_H}\right| = \tan^{-1}\frac{121.24}{30} = 76.10^0$$

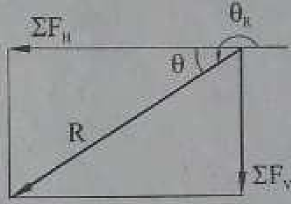


Fig. 2.102

Since both ΣF_V and ΣF_H are negative, the resultant is in the third quadrant .

$$\theta_R = 180 + \theta = 180 + 76.10 = 256.10^0$$

Taking moments of forces about A,

$$\Sigma M_A = (40 \sin 60) \times 2 + (80 \sin 60) \times 8 + (20 \sin 60) \times 12 + (20 \cos 60) \times 2$$
$$= 851.38 \text{ Nm (clockwise).}$$

Let x be the distance of line of action of resultant from A.

$$\Sigma M_A = R \times x$$

$$x = \frac{\Sigma M_A}{R} = \frac{851.38}{124.90} = 6.82 \text{ m}$$

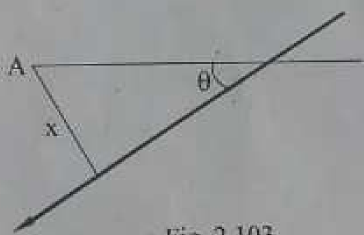


Fig. 2.103

Example 2.38 [KTU May 2019]

For the system of forces shown in Fig. 2.104, determine the magnitude, direction and position of the resultant force with respect to A.

Solution

Resolving the forces horizontally,

$$\Sigma F_H = 100 - 100 + 150 \cos 45$$
$$= 106.07 \text{N}$$

Resolving the forces vertically,

$$\Sigma F_V = 200 - 150 \sin 45$$
$$= 93.93 \text{N}$$

Resultant, $R = \sqrt{\Sigma F_H^2 + \Sigma F_V^2}$

$= \sqrt{106.07^2 + 93.93^2}$

$= 141.68N$

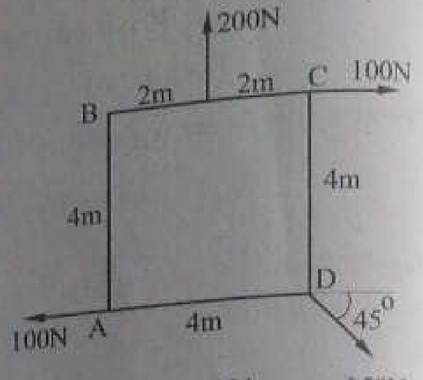


Fig. 2.104

Inclination of resultant with horizontal,

$\theta = \tan^{-1}\frac{\Sigma F_V}{\Sigma F_H}$

$= \tan^{-1}\frac{93.93}{106.07}$

$= 41.53^\circ$

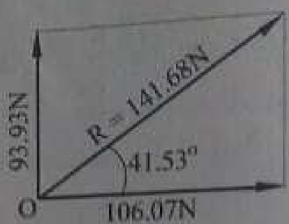


Fig. 2.105

Sum of moments of all the forces about A,

$\Sigma M_A = 150\sin 45 \times 4 + 100 \times 4 - 200 \times 2$

$= 424.26Nm$ cw

Let the perpendicular distance of line of action of resultant from A be x.

Moment of resultant about $A = R \times x$

$= 141.68x$

Equating ΣM_A and $R \times x$,

$424.26 = 141.68x$

$x = 3m.$

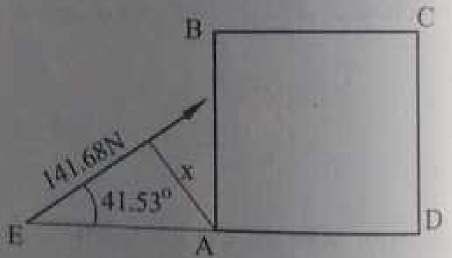


Fig. 2.106

$\frac{x}{AE} = \sin\theta$

Distance $AE = \frac{x}{\sin\theta} = \frac{3}{\sin 41.53}$

$= 4.52m$

2.16. Equilibrium equations of coplanar force system.

For the equilibrium of coplanar force system, the resultant force must be zero and the sum of moment of all the forces about any point in the plane of the forces should also be zero.

Resultant, $R = \sqrt{(\Sigma F_H)^2 + (\Sigma F_V)^2}$

For R to be zero both ΣF_H and ΣF_V should be zero.

$$\Sigma F_H = 0$$

$$\Sigma F_V = 0$$

$\Sigma M = 0$. These are the equations of equilibrium of coplanar force system.

Example 2.39

A slender bar AB of weight 100 N and length 6 m rests on a small roller at D. The end A rests on a smooth vertical wall as shown in Fig.2.107. Calculate the angle θ that the bar makes with the horizontal for the condition of equilibrium.

Solution.

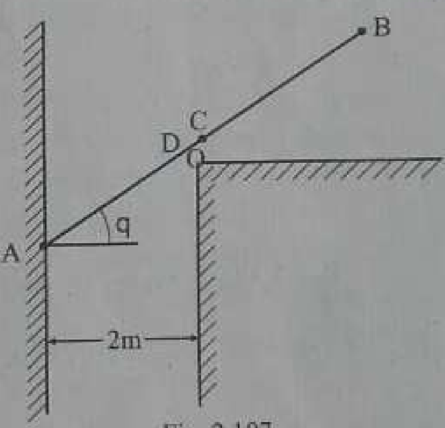


Fig. 2.107

For $\sum F_H = 0$,

$$R_A - R_D \sin\theta = 0 \text{ -------(i)}$$

For $\sum F_V = 0$,

$$R_D \cos\theta - 100 = 0$$

$$R_D = \frac{100}{\cos\theta} \text{ ------(ii)}$$

For $\sum M = 0$, taking moments about A,

$$-R_D \times AD + W\cos\theta \times AC = 0$$

$$-R_D \times \frac{2}{\cos\theta} + 100\cos\theta \times 3 = 0$$

$$R_D = \frac{100\cos\theta \times 3\cos\theta}{2} = 150\cos^2\theta$$

$$\frac{100}{\cos\theta} = 150\cos^2\theta$$

$$150\cos^2\theta\cos\theta = 100$$

$$\cos^3\theta = \frac{100}{150}$$

$$\cos\theta = 0.874$$

$$\theta = 29.07^\circ$$

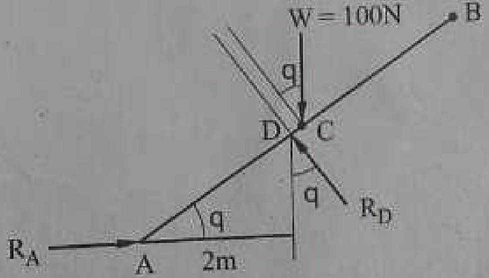


Fig. 2.108

Example 2.40.

A semicircular disc of radius R and weight W rests on a horizontal surface and is pulled at right angles to its geometric axis by a horizontal force P applied as shown in Fig 2.108. Find the angle a that the flat surface of the disc will make with the horizontal just before sliding begins. Coefficient of friction is m. The weight W is acting at the centre of gravity.

Solution.

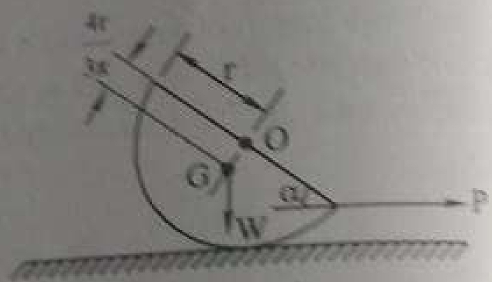


Fig. 2.109

For $\sum F_v = 0$

$$R_N - W = 0$$

$$R_N = W$$

For $\sum F_H = 0$

$$P - \mu R_N = 0$$

$$P - \mu W = 0$$

$$P = \mu W$$

For $\sum M = 0$, taking moments about O,

$$\mu R_N \times r - W \times OG \sin\alpha - P \times r \sin\alpha = 0$$

$$\mu W \times r = W \times \frac{4r}{3\pi} \sin\alpha + \mu W r \sin\alpha = 0$$

$$\mu = \left[\frac{4}{3\pi} + \mu\right] \sin\alpha$$

$$= \frac{4 + 3\pi}{3\pi} \mu \sin\alpha$$

$$\sin\alpha = \frac{3\pi\mu}{3\pi\mu + 4}$$

$$\alpha = \sin^{-1}\left[\frac{3\pi\mu}{3\pi\mu + 4}\right]$$

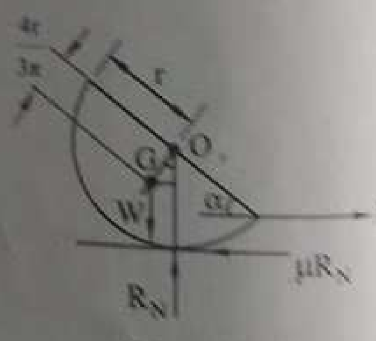


Fig. 2.110